



Planetary Physics (10 points)

Part A. Mid-ocean ridge (5.0 points)

A.1 (0.8 points)

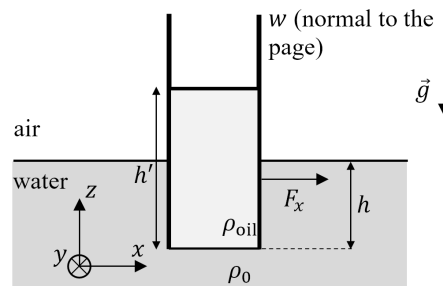


Figure 1

Let h' be the height of the column of oil (see Fig. 1). Then pressure at depth h below the water surface must be $p_h = \rho_{\text{oil}}gh = \rho_{\text{oil}}gh'$, from where $h' = \frac{\rho_0}{\rho_{\text{oil}}}h$. Horizontal force on the plate $F_x = F_1 - F_0$, where the force due to new fluid is $F_1 = \frac{\rho_{\text{oil}}gh'}{2} \cdot h'w$ and the force due to water is $F_0 = \frac{\rho_0gh}{2} \cdot hw$.

Combining all the equation above, we get

$$F_x = \left(\frac{\rho_0}{\rho_{\text{oil}}} - 1 \right) \frac{\rho_0gh^2w}{2}.$$

A.1 (0.8 pt)

$$F_x = \left(\frac{\rho_0}{\rho_{\text{oil}}} - 1 \right) \frac{\rho_0gh^2w}{2}.$$

A.2 (0.6 points)

Consider a rectangular mass element of the crust. Since relation $l(T) = l_1 [1 - k_l (T_1 - T) / (T_1 - T_0)]$ holds for all three dimensions of the solid, its volume V satisfies

$$V = V_1 \left(1 - k_l \frac{T_1 - T}{T_1 - T_0} \right)^3,$$

where V_1 is the volume at $T = T_1$. If the mass of the element is m , density is then

$$\rho(T) = \frac{m}{V} = \frac{m}{V_1} \left(1 - k_l \frac{T_1 - T}{T_1 - T_0} \right)^{-3} = \rho_1 \left(1 - k_l \frac{T_1 - T}{T_1 - T_0} \right)^{-3}.$$



Since $k_l \ll 1$, this can be approximated as

$$\rho(T) \approx \rho_1 \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0} \right),$$

so that $k = 3k_l$.

A.2 (0.6 pt)

$$\rho(T) \approx \rho_1 \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0} \right). \quad k = 3k_l.$$

A.3 (1.1 points)

Since mantle behaves like a fluid in hydrostatic equilibrium, pressure $p(x, z)$ at $z = h + D$ must be the same for all x . Therefore,

$$p(0, h + D) = p(\infty, h + D).$$

Similarly, we must have

$$p(0, 0) = p(\infty, 0).$$

Hence, the change in pressure between $z = 0$ and $z = \infty$ must be the same at both $x = 0$ and $x = \infty$. At the ridge axis

$$p(0, h + D) - p(0, 0) = \rho_1 g (h + D),$$

while far away

$$p(\infty, h + D) - p(\infty, 0) = \rho_0 g h + \int_h^{h+D} \rho(T(\infty, z)) g dz.$$

Far away from the ridge axis the two surfaces of the crust are effectively horizontal, meaning that the law of heat conduction can be written as

$$\frac{dT}{dz} = \text{const.}$$

Hence, after applying the relevant temperature boundary conditions,

$$T(\infty, z) = T_0 + (T_1 - T_0) \frac{z - h}{D}.$$

From all the equations above and by using the density formula given in the problem text,

$$\rho_1 g (h + D) = \rho_0 g h + \int_h^{h+D} \rho_1 \left(1 + k \frac{T_1 - T_0 - (T_1 - T_0) \frac{z-h}{D}}{T_1 - T_0} \right) g dz,$$

from where we straightforwardly obtain

$$D = \frac{2}{k} \left(1 - \frac{\rho_0}{\rho_1} \right) h.$$



A.3 (1.1 pt)

$$D = \frac{2}{k} \left(1 - \frac{\rho_0}{\rho_1} \right) h.$$

A.4 (1.6 points)

The net horizontal force on the half of the ridge is the difference between the pressure forces acting at $x = 0$ and $x = \infty$:

$$F = L \int_0^{h+D} (p(0, z) - p(\infty, z)) dz.$$

From considerations of the previous question, pressure at $x = 0$ is

$$p(0, z) = p(0, 0) + \rho_1 g z,$$

while very far away

$$p(\infty, z) = \begin{cases} p(\infty, 0) + \rho_0 g z & \text{if } 0 \leq z \leq h, \\ p(\infty, 0) + \rho_0 g h + \int_h^z \rho_1 \left(1 + k \frac{T_1 - T_0 - (T_1 - T_0) \frac{z' - h}{D}}{T_1 - T_0} \right) g dz' & \text{if } h \leq z \leq h + D. \end{cases}$$

The equations above can be combined into

$$F = L \int_0^{h+D} (p(0, 0) + \rho_1 g z) dz - L \int_0^h (p(\infty, 0) + \rho_0 g z) dz - \\ - L \int_h^{h+D} (p(\infty, 0) + \rho_0 g h) dz - L \int_h^{h+D} \left[\int_h^z \rho_1 \left(1 + k \left(1 - \frac{z' - h}{D} \right) \right) g dz' \right] dz.$$

The double integral can be easily found either directly or by using a substitution $u = z - h$, $u' = z' - h$:

$$\int_h^{h+D} \left[\int_h^z \rho_1 \left(1 + k \left(1 - \frac{z' - h}{D} \right) \right) g dz' \right] dz = \int_0^D \left[\int_0^u \rho_1 \left(1 + k \left(1 - \frac{u}{D} \right) \right) g du' \right] du$$

After a straightforward integration and using $p(0, 0) = p(\infty, 0)$ as well as the result of the previous question,

$$F = gL \left[\rho_1 \left(\frac{h^2}{2} + hD - \frac{kD^2}{3} \right) - \rho_0 \left(\frac{h^2}{2} + hD \right) \right] = gLh^2 (\rho_1 - \rho_0) \left(\frac{1}{2} + \frac{2}{3k} \left(1 - \frac{\rho_0}{\rho_1} \right) \right).$$

Since $k \ll 1$, the term with $\frac{1}{k}$ is of the leading order, hence, the required answer is

$$F \approx \frac{2gLh^2 (\rho_1 - \rho_0)^2}{3k \rho_1}.$$



A.4 (1.6 pt)

$$F \approx \frac{2gLh^2}{3k} \frac{(\rho_1 - \rho_0)^2}{\rho_1}.$$

A.5 (0.9 points)

The timescale τ is expected to depend only on density of the crust ρ_1 , its specific heat c , thermal conductivity κ and thickness D . Hence, we can write $\tau = A\rho_1^\alpha c^\beta \kappa^\gamma D^\delta$, where A is a dimensionless constant. We will obtain the powers α – δ via dimensional analysis.

Define the symbols for different dimensions: L for length, M for mass, T for time and Θ for temperature. Then τ , ρ_1 , c , κ and D have dimensions T, ML^{-3} , $\text{L}^2\text{T}^{-2}\Theta^{-1}$, $\text{MLT}^{-3}\Theta^{-1}$ and L, respectively. The resulting set of linear equations to balance the powers of length, mass, time and temperature, respectively, is

$$\begin{cases} 0 = -3\alpha + 2\beta + \gamma + \delta, \\ 0 = \alpha + \gamma, \\ 1 = -2\beta - 3\gamma, \\ 0 = -\beta - \gamma. \end{cases}$$

This gives $\alpha = \beta = 1$, $\gamma = -1$, $\delta = 2$. Hence,

$$\tau = A \frac{c\rho_1 D^2}{\kappa}.$$

A.5 (0.9 pt)

$$\tau \approx \frac{c\rho_1 D^2}{\kappa}.$$

Part B. Seismic waves in a stratified medium (5.0 points)

B.1 (1.5 points)

Seismic waves in this problem can be treated by using ray theory. Namely, their propagation is described by the Snell's law of refraction

$$n(0) \sin \theta_0 = n(z) \sin \theta,$$

where the refractive index is

$$n(z) = \frac{c}{v(z)} = \frac{c}{v_0 \left(1 + \frac{z}{z_0}\right)}$$

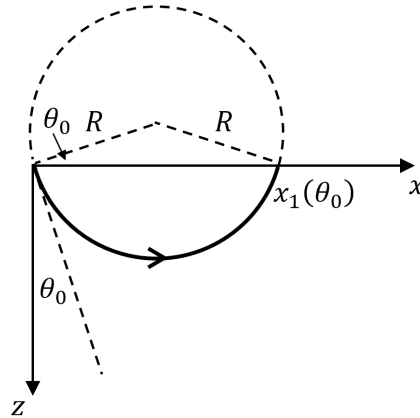


Figure 2

and c denotes the seismic wave speed in a material with refractive index $n = 1$. From the two equations above we have

$$v_0 \left(1 + \frac{z}{z_0} \right) \sin \theta_0 = v_0 \sin \theta.$$

Method 1. Since this describes an arc of a circle, we have that at $\theta = \frac{\pi}{2}$, $z = R - R \sin \theta_0$ (fig. 2), giving

$$\left(1 + \frac{R - R \sin \theta_0}{z_0} \right) \sin \theta_0 = 1,$$

from where the circle radius $R = \frac{z_0}{\sin \theta_0}$. From simple geometry we get

$$x_1(\theta_0) = 2R \cos \theta_0,$$

i.e. $A = 2z_0$ and $b = 1$.

Method 2. Implicitly differentiating $v_0 \left(1 + \frac{z}{z_0} \right) \sin \theta_0 = v_0 \sin \theta$ gives

$$\frac{dz}{z_0} \sin \theta_0 = \cos \theta d\theta.$$

An infinitesimal ray path length dl is related to the change in the vertical coordinate via

$$dz = dl \cos \theta,$$

giving

$$dl = \frac{z_0}{\sin \theta_0} d\theta.$$

This is an equation of an arc of a circle of radius $R = \frac{z_0}{\sin \theta_0}$

Alternatively, instead of considering an infinitesimal ray path length dl , one can obtain the answer by writing

$$\cot \theta = \frac{dz}{dx} = \frac{dz d\theta}{d\theta dx}.$$

The first derivative can be eliminated via Snell's law, leading to

$$\cot \theta = \frac{z_0 \cos \theta}{\sin \theta_0} \frac{d\theta}{dx},$$

which can be integrated to get

$$x_1 = -\frac{z_0}{\sin \theta_0} \int_{\text{start}}^{\text{end}} d\cos \theta = \frac{2z_0 \cos \theta_0}{\sin \theta_0},$$

where we used Snell's law again to get that the ray has $\cos \theta = -\cos \theta_0$ at the point where it reaches the surface.

B.1 (1.5 pt)

$$x_1(\theta_0) = 2z_0 \cot \theta_0.$$

B.2 (1.5 points)

In two dimensions, $\frac{E}{\pi} d\theta_0$ is the energy carried by rays that are emitted within interval $[\theta_0, \theta_0 + d\theta_0]$. On the other hand, the energy carried by rays that arrive at $[x, x + dx]$ is εdx . Therefore,

$$\varepsilon = \frac{E}{\pi} \left| \frac{d\theta_0}{dx} \right|.$$

Using the result of question B.1,

$$\frac{dx}{d\theta_0} = -\frac{Ab}{\sin^2(b\theta_0)} = -Ab(1 + \cot^2(b\theta_0)) = -\frac{b(A^2 + x^2)}{A}.$$

Hence,

$$\varepsilon(x) = \frac{EA}{\pi b(A^2 + x^2)} = \frac{2Ez_0}{\pi(4z_0^2 + x^2)}.$$

This function is plotted in Fig. 3.

B.2 (1.5 pt)

$$\varepsilon(x) = \frac{EA}{\pi b(A^2 + x^2)} = \frac{2Ez_0}{\pi(4z_0^2 + x^2)}.$$

Sketch is shown in Fig. 3.

B.3 (2.0 points)

Define $x_- = x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right)$ and $x_+ = x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right)$. To the leading order in $\delta\theta_0$, $x_- \approx x_+ \approx x_1(\theta_0)$. With each reflection of the signal, the horizontal distance between the points where the edges of

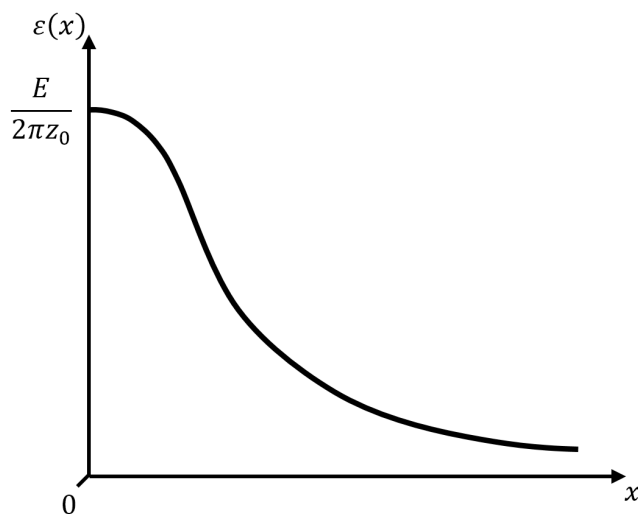


Figure 3. Plot of the function $\varepsilon(x)$.

the signal reflect increases by $|x_+ - x_-| = x_- - x_+$. When moving along the positive x -axis, these zones get wider until they overlap. If this happens after N reflections, then

$$N \approx \frac{x_1(\theta_0)}{x_- - x_+},$$

where the approximate sign tends to equality as $\delta\theta_0 \rightarrow 0$.

The position where the zones start to overlap is at $x_{\max} = Nx_1(\theta_0)$. Therefore,

$$x_{\max} = \frac{x_1(\theta_0)^2}{x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right) - x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right)}.$$

Since $\delta\theta_0 \ll \theta_0$, we can approximate

$$x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right) - x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right) \approx -\frac{dx_1(\theta_0)}{d\theta_0} \delta\theta_0 = \frac{Ab}{\sin^2(b\theta_0)} \delta\theta_0.$$

Combining the last two equations and substituting the $x_1(\theta_0)$ expression gives

$$x_{\max} = \frac{Ab \cos^2(b\theta_0)}{\delta\theta_0} = \frac{2z_0 \cos^2 \theta_0}{\delta\theta_0}.$$

B.3 (2.0 pt)

$$x_{\max} = \frac{Ab \cos^2(b\theta_0)}{\delta\theta_0} = \frac{2z_0 \cos^2 \theta_0}{\delta\theta_0}.$$