A Mechanical Model for Phase Transitions

General Grading Guidelines

When student's solutions are correct and s/he also show how solutions were obtained, the stduent gets full credit. The scheme oulined below is helpful if the student's answers are partially correct. Attention will be paid to the detailed solution so, if the final answer is correct but it is obtained by incorrect method(s) then no credit will be given. Alternative solutions may exist and will be given due credit.

Partial or full outcomes obtained for later sections in the problem which are incorrect solely because of errors being carried forward from previous sections, but are otherwise reasonable, will not be further penalized. For example a dimensioanlly wrong answer when carried forward will not get any credit in the subsequent sections. A numerically wrong evaluation when carried forward will get credit in subsequent sections unless the numerical answer is patently wrong (e.g. the value of g is 981 m/sec^2!)

Incorrect or no labeling of an axis is penalized by -0.1 points

The numerical answer (i) must be correct to +/- 10% AND (ii) must respect significant figures.

It maybe noted that NO micro-marking scheme takes care of all contingencies. A certain amount of discretion rests with and a certain level of judgement is invested in the academic committee.

A.1 (0.5 pt)
Equations of motion

The radial component F_r yields:

$$mR\dot{\theta}^2 = N - mg\cos(\theta) - mR\sin^2(\theta)\omega^2 \quad (1)$$

[0.2]

The tangential component F_θ

$$mR\ddot{\theta} = mR\sin(\theta)\cos(\theta)\omega^2 - mg\sin(\theta) - \mathrm{sgn}\left(\dot{\theta}\right)kN \quad (2)$$

OR

$$mR\ddot{\theta} = mR\sin(\theta)\cos(\theta)\omega^2 - mg\sin(\theta) - fkN \quad (f = 1) \quad (3)$$

[0.3]

B.1 (1.0 pt)
Equilibrium angle(s)

We set $k = 0$ in the equation for the tangential component of the force. Thus

$$mR\ddot{\theta} \quad = \quad mR\sin(\theta)\cos(\theta)\omega^2 - mg\sin(\theta) \tag{4}$$

For equilibrium we set $\ddot{\theta}_0 = 0$ in the above equation. Then $\theta_0 = 0$ is an equilibrium angle for all values of ω

[0.4]

The other values are given by

$$\cos\theta_0 = \frac{g}{\omega^2 R} = \frac{\omega_c^2}{\omega^2} \tag{5}$$

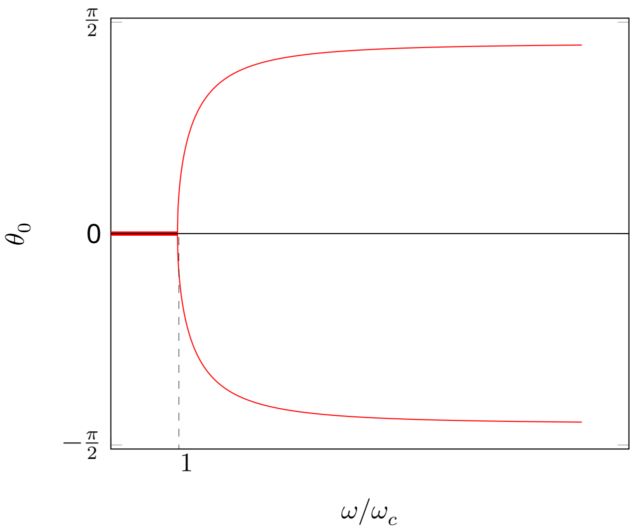
OR

$$\theta_0 = \pm\left|\cos^{-1}\frac{\omega_c^2}{\omega^2}\right| \tag{6}$$

[0.6]

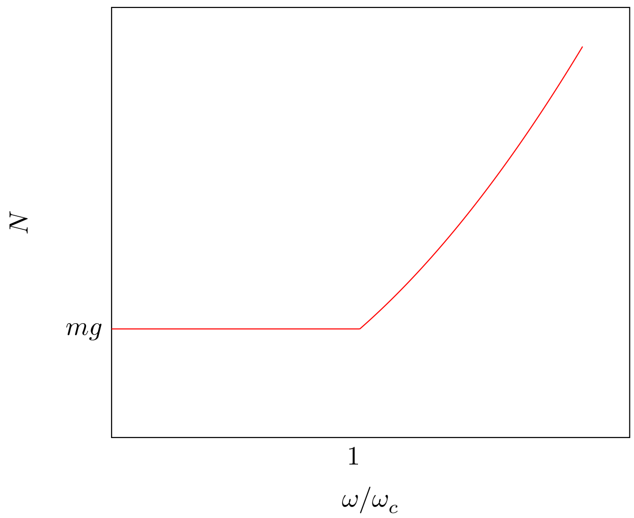
with values of θ_0 between -$\pi/2$ to $\pi/2$. The $\pm$ indicates that there are two equivalent positions. The bead could rise on either side of the axis shown in the figure depicted in the problem. Note that for $\omega < \omega_c$, Eq. (5) implies $\cos\theta_0 > 1$. This is clearly unphysical. A little reflection will convince us that $\theta_0 = 0$ for $\omega < \omega_c$.

B.2 (0.5 pt)
Sketch of θ_0.



[shape correct: 0.2]
[only if both branches: 0.3]

B.3 (0.5 pt)
Sketch of the magnitude of the normal reaction



[$\omega < \omega_c$: **0.2**]
[$\omega > \omega_c$: **0.3**]

B.4 (1.0 pt)
The potential $V(\theta)$

Given that

$$F_\theta = -\frac{1}{R}\frac{dV(\theta)}{d\theta} \tag{7}$$

% and taking $V(\theta = 0) = 0$, we obtain on integrating Eq. (4) that

$$-R\int_0^\theta F_\theta d\theta = \int_0^V dV = V - 0 \tag{8}$$

[0.3]

the left hand side is

$$\begin{aligned} -R\int_0^\theta F_\theta d\theta &= \frac{-m\omega^2 R^2}{2}\int_0^\theta \sin(2\theta) + mgR\int_0^\theta \sin(\theta)d\theta \\ &= \frac{m\omega^2 R^2(\cos(2\theta)-1)}{4} - mgR(\cos(\theta)-1) \end{aligned} \tag{9}$$

[0.4]

Noting that $cos(2(\theta)-1) = -2\sin^2(\theta)$ and $\omega_c^2 = g/R$ we obtain

$$V(\theta) = mgR\left[(1-\cos\theta) - \frac{\omega^2}{2\omega_c^2}\sin^2\theta\right] \tag{10}$$

[0.3]

We can also verify the above equation by substitution into Eq. (7)
$P = mgR$
$Q = -mgR$
$S = -\dfrac{\omega^2 mgR}{2\omega_c^2}$

B.5 (1.0 pt)
The coefficients

We use the expansions for the trigonmetric functions sin(θ) and cos(θ) in Eq.(10). We shall keep terms upto and inculding order θ^4. Thus

$$\begin{aligned} V(\theta) &\approx mgR\left[1 - 1 + \theta^2/2 - \theta^4/24 - \frac{\omega^2}{2\omega_c^2}(\theta - \theta^3/6)^2\right] \\ &\approx \frac{mgR}{2}\left[1 - \frac{\omega^2}{\omega_c^2}\right]\theta^2 + \frac{mgR}{6}\left[\frac{\omega^2}{\omega_c^2} - \frac{1}{4}\right]\theta^4 \end{aligned}$$

Thus

$$a(\omega) = \frac{mgR}{2}(1 - \frac{\omega^2}{\omega_c^2})$$

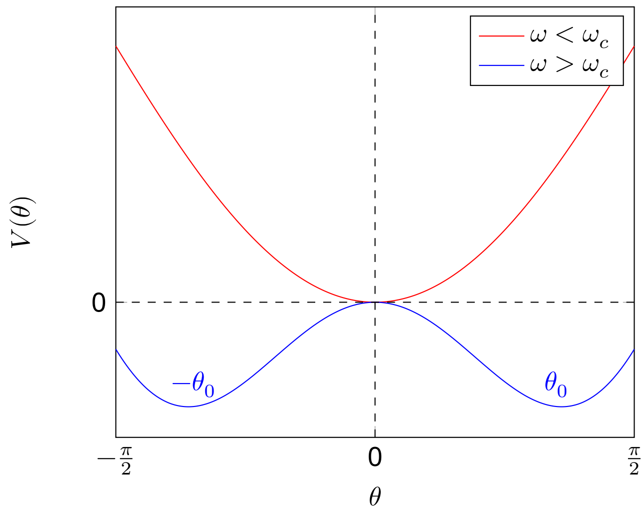
[0.5]

$$b(\omega) = \frac{mgR}{6}(\frac{\omega^2}{\omega_c^2} - \frac{1}{4})$$

[0.5]

Note: no penalty if the 1/4 term is missed.

B.6 (1.0 pt)
Representative plots of the potential



[$\omega < \omega_c$: 0.5]
[$\omega > \omega_c$: 0.5]

B.7 (1.0 pt)
Bead analogues
For $\omega \to \omega_c^+$, θ_0 is close to zero. Hence on expanding the cosine term in Eq. (5),

$$1 - \frac{\theta_0^2}{2} = \frac{\omega_c^2}{\omega^2}$$
$$\theta_0 = \pm\sqrt{2}\left[1 - \frac{\omega_c^2}{\omega^2}\right]^{1/2} \tag{11}$$

Also note from Eq. (5) that as $\omega \to \infty$, $\theta_0 \to \pm\pi/2$. This plot also has an analogue in phase transition. The magnetization $\mathcal{M}$ goes to zero as T goes to T_c in a similar fashion. Thus the role of $\mathcal{M}$ is played by θ_0 and temperature is inversely related to ω. Increasing temperature is equivalent to decreasing ω. Summarizing,

$$\mathcal{M} \longrightarrow \theta$$

[0.4]

$$T_c \longrightarrow 1/\omega_c^2$$
$$T/T_c \longrightarrow \omega_c^2/\omega^2$$

[0.4]

Equivalent value of β for bead is = 1/2.

[0.2]

[Note: The critical exponent is $1/2$ in our case and also in Landau theory. However experimentally and in more elaborate theories the exponent of vanishing magnetization is $1/3$].

B.8 (1.0 pt)
Oscillation frequency

The frequency of oscillation Ω_0 of the bead about the "equilibrium" position θ_0 is

$$\Omega_0 = \frac{1}{R}\sqrt{\frac{V''(\theta)}{m}}$$

We take the second order derivative of the potential as given in Eq. (10)

$$V''(\theta) = mgR\cos\theta\left[1 - \frac{\omega^2}{\omega_c^2}\cos\theta\right] + mgR\,\frac{\omega^2}{\omega_c^2}\,\sin^2\theta \tag{12}$$

For $\theta = \theta_0 = \pm\cos^{-1}\left(\omega_c^2/\omega^2\right)$

$$V''(\theta_0) = mgR\,\frac{\omega^2}{\omega_c^2}\left(1 - \frac{\omega_c^4}{\omega^4}\right) > 0 \qquad \text{if } \omega > \omega_c \tag{13}$$

For $\omega < \omega_c$, $\theta_0 = 0$, and we obtain from Eq. (12) that

$$\Omega_0 = (\omega_c^2 - \omega^2)^{1/2} \tag{14}$$

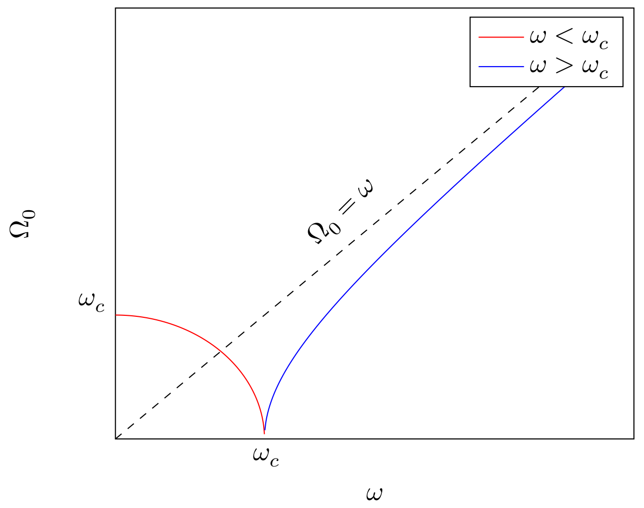
[0.5]

Similarly for $\omega > \omega_c$, using Eq. (13) we obtain

$$\Omega_0 = \omega\left(1 - \frac{\omega_c^4}{\omega^4}\right)^{1/2} \tag{15}$$

[0.5]

B.9 (1.0 pt)
Sketch of Ω_0



[$\omega < \omega_c$: **0.5**]
[$\omega > \omega_c$: **0.5**]

C.1 (1.0 pt)
Condition for equilibrium angles

We substitute the expression for the normal reaction (Eq.(1)) in the angular part (Eq.(3)) to obtain

$$mR\ddot{\theta} \quad = \quad mR\sin(\theta)\cos(\theta)\omega^2 - mg\sin(\theta) - fk(mg\cos(\theta) + mR\sin^2(\theta)\omega^2 + mR\dot{\theta}^2)$$

Noting that $\omega_c^2 = g/R$ and rearranging terms we have

$$\ddot{\theta} \quad = \quad \omega_c^2\left[(\sin(\theta))(\cos(\theta) - fk\sin(\theta))\left(\frac{\omega}{\omega_c}\right)^2 - \sin(\theta) - fk\cos(\theta) - fk\left(\frac{\dot{\theta}}{\omega_c}\right)^2\right]$$

[0.2]

At equilibrium, $\dot{\theta} = 0, \ \ddot{\theta} = 0$ and $f = \text{sgn}\left(\dot{\theta}\right) = \pm 1$ depending on how this equilibrium was attained, ie., depending on the value of $\dot{\theta}$ just before equilibrium was attained. Thus we obtain the expression for the equilibrium angle θ_0,

$$\boxed{\sin(\theta_0)\left(\cos(\theta_0) - fk\sin(\theta_0)\right)\left(\frac{\omega}{\omega_c}\right)^2 = \sin(\theta_0) + fk\cos(\theta_0) \text{ with } \theta_0 \in (-\pi/2, \pi/2)}$$

[0.4]

For $f = 1$ and $k = \tan(\alpha)$ we may express the above as

$$\begin{aligned}\left(\frac{\omega}{\omega_c}\right)^2 &= \frac{\sin(\theta_0) + \tan(\alpha)cos(\theta_0)}{\sin(\theta_0)(\cos(\theta_0) - \tan(\alpha)\sin(\theta_0))} \\ &= \frac{\tan(\theta_0 + \alpha)}{\sin(\theta_0)} \qquad (16)\end{aligned}$$

[0.4]

C.2 (0.5 pt)

Representative values for θ_0

We are given the expansions for the trignometric functions in the problem. We notice that the coefficient of friction k is small (=0.05). Thus k = α. We then have

$$\sin(\theta_0) \approx \theta_0$$

$$\tan(\theta_0 + \alpha) \approx \theta_0 + \alpha$$

[0.2]

Thus

$$\left(\frac{\omega}{\omega_c}\right)^2 \approx 1 + \frac{k}{\theta_0}$$

Simple calculations yield

(a) θ_0 = -0.07 radians

(b) θ_0 = -0.1 radians

[0.3]

The plot will no longer be symmetric.