Theory Problem 1: Characterization of Soil Colloids (10 points)

Part A. Analysis of motions of colloidal particles (1.6 points)

A.1 The relation between the impulse and the momentum change is given by $Mv_0 = I_0$. Therefore,

$$v_0 = \frac{I_0}{M}. \tag{S1.1}$$

For the situation considered here, the equation of motion reads

$$M\dot{v} = -\gamma v(t). \tag{S1.2}$$

Substituting the form of the solution given in the question sheet, $v(t) = v_0 e^{-(t-t_0)/\tau}$, we obtain

$$\tau = \frac{M}{\gamma}. \tag{S1.3}$$

A.1	0.8 pt
$v_0 = \frac{I_0}{M}$ $\tau = \frac{M}{\gamma}$	

A.2 Thanks to the linearity of Eq. (S1.2), we can use the superposition principle, which tells us that $v(t)$ is given by the sum of solutions for single collision events that occur before time t. This immediately gives the solution as

$$v(t) = \sum_i \frac{I_i}{M} e^{-(t-t_i)/\tau}, \tag{S1.4}$$

where the sum is taken in the range of i that satisfies $0 < t_i < t$.

It is also not difficult to figure out this superposition principle, by considering the effect of a single collision as well as the velocity change between two consecutive collisions. From A.1, it is straightforward to show that the velocity right after the ith collision is given by

$$v(t_i) = v_0(t_i) + \frac{I_i}{M}, \tag{S1.5}$$

where $v_0(t_i)$ is the velocity right before the collision. Also, since there is no collision during $t_i < t < t_{i+1}$, we have

$$v(t) = \left(v_0(t_i) + \frac{I_i}{M}\right) e^{-(t-t_i)/\tau}. \tag{S1.6}$$

In particular,

$$v_0(t_{i+1}) = \left(v_0(t_i) + \frac{I_i}{M}\right) e^{-(t_{i+1}-t_i)/\tau}. \tag{S1.7}$$

Therefore, with $v_0(t_1) = 0$, we obtain

$$v_0(t_i) = \sum_{j=1}^{i-1} \frac{I_j}{M} e^{-(t_i-t_j)/\tau} \tag{S1.8}$$

and, for $t_i < t < t_{i+1}$,

$$v(t) = \sum_{j=1}^{i} \frac{I_j}{M} e^{-(t-t_j)/\tau}. \tag{S1.9}$$

This is equivalent to Eq. (S1.4).

A.2	0.8 pt
$v(t) = \sum_{i} \frac{I_i}{M} e^{-(t-t_i)/\tau}$ the inequality specifying the range of t_i that needs to be considered: $0 < t_i < t$	

Part B. Effective equation of motion (1.8 points)

B.1 From the definition of the model, we have

$$\Delta x(t) = \sum_{n=1}^{N} v_n \delta. \tag{S1.10}$$

Taking the average and using $\langle v_n \rangle = 0$, we obtain

$$\langle \Delta x(t) \rangle = 0. \tag{S1.11}$$

For the mean square displacement, computing the square of Eq. (S1.10) and taking the average, we obtain

$$\langle \Delta x(t)^2 \rangle = \sum_{m=1}^{N} \sum_{n=1}^{N} \langle v_m v_n \rangle \delta^2. \tag{S1.12}$$

Using $\langle v_m v_n \rangle = C$ for $n = m$ and 0 otherwise, we find

$$\langle \Delta x(t)^2 \rangle = \sum_{n=1}^{N} C\delta^2 = NC\delta^2. \tag{S1.13}$$

Since $N\delta = t$, we obtain

$$\langle \Delta x(t)^2 \rangle = C\delta t. \tag{S1.14}$$

B.1	1.0 pt
$\langle \Delta x(t) \rangle = 0$ $\langle \Delta x(t)^2 \rangle = C\delta t$	

B.2 As described in the question sheet, the mean square displacement $\langle \Delta x(t)^2 \rangle$ is a characteristic observable of the Brownian motion, which of course takes a finite value for a given t. For the model considered here, we have Eq. (S1.14), but we need to consider the limit $\delta \to 0$ to describe the Brownian motion in this model. This requires that $C\delta$ remains finite, so that $C \propto \delta^{-1}$. It also follows that $\langle \Delta x(t)^2 \rangle \propto t$.

▷ Note: The continuous time limit $\delta \to 0$ of the present model corresponds to what is called the overdamped Langevin equation. This reads, in the absence of external force as considered here,

$$\gamma \frac{dx}{dt} = \xi(t) \tag{S1.15}$$

with a Gaussian noise $\xi(t)$ that satisfies

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t)\xi(t') \rangle = 2D\delta(t-t') \tag{S1.16}$$

with the diffusion coefficient D. Here, $\delta(t)$ (not to confuse with δ in the problem) is called the delta function, which satisfies $\delta(t) = 0$ for $t \neq 0$ and $\delta(0) = \infty$ but $\int_a^b \delta(t)dt = 1$ for any $a < 0$ and $b > 0$.

B.2 0.8 pt

$\alpha = -1$

$\beta = 1$

Part C. Electrophoresis (2.7 points)

C.1 For particles with velocity v (> 0), only those in the range $x_0 - v\delta \leq x \leq x_0$ pass the position x_0 during a time interval δ. Therefore, the number of such particles per unit cross-sectional area and per unit time is given by

$$N_+(x_0) = \frac{1}{\delta}\int_{x_0 - v\delta}^{x_0} \frac{1}{2} n(x)dx \tag{S1.17}$$

Using the Taylor expansion $n(x) \simeq n(x_0) + (x - x_0)\frac{dn}{dx}(x_0)$ and integrating, we obtain

$$N_+(x_0) = \frac{1}{2}n(x_0)v - \frac{1}{4}\frac{dn}{dx}(x_0)v^2\delta. \tag{S1.18}$$

C.1 0.5 pt

$$N_+(x_0) = \frac{1}{2}n(x_0)v - \frac{1}{4}\frac{dn}{dx}(x_0)v^2\delta$$

C.2 Let $N_-(x_0)$ be the counterpart of $N_+(x_0)$ for particles with velocity $-v$, then

$$N_-(x_0) = \frac{1}{2}n(x_0)v + \frac{1}{4}\frac{dn}{dx}(x_0)v^2\delta. \tag{S1.19}$$

With this equation, Eq. (S1.18), and $J_D(x_0) = \langle N_+(x_0) - N_-(x_0) \rangle$, we obtain

$$J_D(x_0) = -\frac{1}{2}\frac{dn}{dx}(x_0)\langle v^2 \rangle \delta = -\frac{1}{2}\frac{dn}{dx}(x_0)C\delta. \tag{S1.20}$$

Comparing this with Eq. (4) in the question sheet for $x = x_0$, $J_D(x_0) = -D\frac{dn}{dx}(x_0)$, we obtain

$$D = \frac{1}{2}C\delta. \tag{S1.21}$$

Plugging this into the result of B.1, we obtain

$$\langle \Delta x(t)^2 \rangle = 2Dt. \tag{S1.22}$$

C.2 0.7 pt

$$J_D(x_0) = -\frac{1}{2}\frac{dn}{dx}(x_0)C\delta$$

$$D = \frac{1}{2}C\delta$$

$$\langle \Delta x(t)^2 \rangle = 2Dt$$

C.3 The force balance sketched in Fig. 2 is expressed by the following equation:

$$\Pi(x)A + n(x)A\Delta x QE = \Pi(x+\Delta x)A. \quad \text{(S1.23)}$$

Using the van 't Hoff equation for the osmotic pressure, $\Pi(x) = n(x)kT$, and carrying out the Taylor expansion of $n(x+\Delta x)$, we obtain

$$\frac{dn}{dx} = \frac{n(x)}{kT}QE. \quad \text{(S1.24)}$$

C.3 0.5 pt

$$\frac{dn}{dx} = \frac{n(x)}{kT}QE$$

C.4 The equation of motion for $\langle v(t) \rangle$ is

$$M\frac{d\langle v(t)\rangle}{dt} = -\gamma\langle v(t)\rangle + QE. \quad \text{(S1.25)}$$

By solving this with the initial condition $\langle v(0)\rangle = 0$, we obtain

$$\langle v(t)\rangle = \frac{QE}{\gamma}(1 - e^{-t/\tau}). \quad \text{(S1.26)}$$

Therefore,

$$u = \lim_{t\to\infty}\langle v(t)\rangle = \frac{QE}{\gamma}. \quad \text{(S1.27)}$$

▷ Note: The student is expected to surmise that the solution to Eq. (S1.25) has a functional form analogous to that to Eq. (S1.2), whose solution is given in the question sheet.

C.4 0.5 pt

$$\langle v(t)\rangle = \frac{QE}{\gamma}(1 - e^{-t/\tau})$$

$$u = \frac{QE}{\gamma}$$

C.5 From the result of C.3 and Eq. (4) in the question sheet, we have

$$J_D(x) = -\frac{DQE}{kT}n(x). \quad \text{(S1.28)}$$

From the result of C.4 and Eq. (5) in the question sheet, we have

$$J_Q(x) = \frac{QE}{\gamma} n(x). \tag{S1.29}$$

Plugging these into the flux balance condition, $J_D(x) + J_Q(x) = 0$, we obtain

$$D = \frac{kT}{\gamma}. \tag{S1.30}$$

C.5 0.5 pt

$$D = \frac{kT}{\gamma}$$

Part D. Mean square displacement (2.4 points)

D.1 Combining the results of C.2 and C.5, $k = R/N_A, \gamma = 6\pi a\eta$, we obtain the following equation that links the mean square displacement to N_A:

$$\langle \Delta x^2 \rangle = \frac{RT\Delta t}{3\pi a\eta N_A}. \tag{S1.31}$$

From the data given in the question sheet, the mean square displacement is estimated at $\langle \Delta x^2 \rangle = 6.34\ \mu\mathrm{m}^2$. Plugging this and the values of the parameters given in the question sheet, we obtain

$$N_A = 5.6 \times 10^{23}\ \mathrm{mol}^{-1}. \tag{S1.32}$$

▷ Note: In 1908, Jean Baptiste Perrin (1870-1942) carried out such an observation and obtained an estimate of N_A, which turned out to be consistent with the values known at that time by other approaches. This convinced the community of the fact that molecules and hence atoms do exist as constituents of matter. Perrin was awarded the Nobel Prize in Physics in 1926 for ``his work on the discontinuous structure of matter, and especially for his discovery of sedimentation equilibrium''. For more details, see, e.g., S. G. Brush, ``A History of Random Processes: I. Brownian Movement from Brown to Perrin'', Archive for History of Exact Sciences, volume **5**, pages 1–36 (1968).

▷ Note: On May 20, 2019, the definition of physical constants including the Avogadro constant N_A was changed. As a result, N_A is now defined by a fixed value, not to be determined through measurements.

D.1 1.0 pt

$$N_A = 5.6 \times 10^{23}\ \mathrm{mol}^{-1}$$

D.2 Using $\Delta x(t) = \sum_{n=1}^{N} (u + v_n)\delta$ and Eq. (3) in the question sheet, we obtain

$$\langle \Delta x^2 \rangle = (ut)^2 + 2Dt \tag{S1.33}$$

for general t. This can be rewritten as

$$\langle \Delta x^2 \rangle = u^2 t \left(t + \frac{2D}{u^2} \right) = u^2 t (t + t_*), \tag{S1.34}$$

with $t_* = 2D/u^2$. Therefore,

$$\langle \Delta x^2 \rangle \propto \begin{cases} t & \text{for } t \ll t_* \\ t^2 & \text{for } t \gg t_* \end{cases} \tag{S1.35}$$

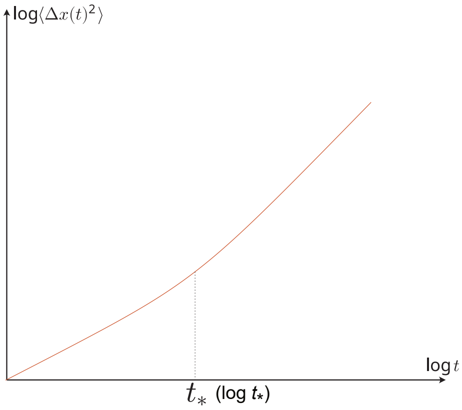
D.2 0.8 pt

$\langle \Delta x^2 \rangle = (ut)^2 + 2Dt$ for general t

$$\langle \Delta x^2 \rangle \propto \begin{cases} t & \text{for small } t \\ t^2 & \text{for large } t \end{cases}$$

$$t_* = \frac{2D}{u^2}$$

An example of the graph to answer:



D.1 Since the microbe does not change the swimming direction for $t \ll \delta_0$, we can use the result of D.2 just by replacing u by u_0. By contrast, for $t \gg \delta_0$, the motion of the microbe can be described by the model considered in PART B, though its parameter δ is not an artificial parameter anymore but is now a quantity that characterizes the microbe's motion, δ_0. The parameter C is given by $C = u_0^2$. Plugging this into Eq. $(\mathrm{S1.14})$ and collecting all these results, we obtain

$$\langle \Delta x^2 \rangle = \begin{cases} 2Dt & \text{for } t \ll 2D/u_0^2 \\ u_0^2 t^2 & \text{for } 2D/u_0^2 \ll t \ll \delta_0 \\ u_0^2 \delta_0 t & \text{for } \delta_0 \ll t \end{cases} \quad (\mathrm{S1.36})$$

▷ Note: More precisely, one can show $\langle \Delta x^2 \rangle = (u_0^2 \delta_0 + 2D)t$ for $t \gg \delta_0$. However, in order for the intermediate regime to exist, we have $2D/u_0^2 \ll \delta_0$, from which it follows that $u_0^2 \delta_0 \gg 2D$ and the expression in Eq. $(\mathrm{S1.36})$ is a good approximation.

▷ Note: The motion of the microbe described here is called the run-and-tumble motion, except that it is usually assumed that the microbe changes the swimming direction (``tumbling'') at random time intervals. Some bacteria including *Escherichia coli* is known to swim in this manner.

D.3 0.6 pt

$$\langle \Delta x^2 \rangle = \begin{cases} 2Dt & \text{for small } t \\ u_0^2 t^2 & \text{for intermediate } t \\ u_0^2 \delta_0 t & \text{for large } t \end{cases}$$

Part E. Water purification (1.5 points)

E.1 The interaction energy $U(d)$ has a barrier if c is small enough, but the barrier disappears if c exceeds a threshold. This threshold is the critical concentration to derive in this question. The condition for the barrier to disappear is given by

$$\min U'(d) = 0. \quad (\mathrm{S1.37})$$

This can be expressed by the following two equations:

$$U'(d) = \frac{A}{d^2} - \frac{B\epsilon(kT)^2}{q^2\lambda} e^{-d/\lambda} = 0, \tag{S1.38}$$

$$U''(d) = -\frac{2A}{d^3} + \frac{B\epsilon(kT)^2}{q^2\lambda^2} e^{-d/\lambda} = 0. \tag{S1.39}$$

Solving these, we obtain

$$d = 2\lambda = \sqrt{\frac{Aq^2\lambda}{B\epsilon(kT)^2}} \tag{S1.40}$$

and therefore

$$\lambda = \frac{e^2 Aq^2}{4B\epsilon(kT)^2}. \tag{S1.41}$$

Plugging this into $c = \dfrac{\epsilon kT}{2N_A q^2}\lambda^{-2}$, we obtain

$$c = \frac{8B^2\epsilon^3(kT)^5}{e^4 N_A A^2 q^6}. \tag{S1.42}$$

▷ Note: In the literature, it is also common to consider that the critical concentration is reached when the energy barrier becomes as low as the energy for $d \to \infty$, i.e., $\max U(d) = 0$, although this does not meet the requirements given in the question sheet. If this condition is used instead, we find $c = \frac{B^2\epsilon^3(kT)^5}{2e^2 N_A A^2 q^6}$. This differs from Eq. (S1.42) only by a factor $e^2/8 \approx 0.92$.

E.1 $c = \dfrac{8B^2\epsilon^3(kT)^5}{e^4 N_A A^2 q^6}$	1.5 pt

Theory Problem 2: Neutron Stars (10 points)

Part A. Mass and stability of nuclei (2.5 points)

A.1 The given binding energy is often called the Weizsäcker-Bethe mass formula, and the physical interpretation of the volume and the surface terms is based on the liquid drop model. The formula works quite well except for the shell effects. Find A to minimize the binding energy per mass number, i.e.,

$$\frac{B}{A} = a_V - a_S A^{-1/3} - \frac{a_C}{4} A^{2/3} . \tag{S2.1}$$

As long as A is small, the second term is dominantly increasing with increasing A, and it is eventually taken over by the third term which is decreasing. Therefore, the extremal corresponds to the maximum of B/A. One can explicitly carry out

$$\frac{d(B/A)}{dA} = 0 \tag{S2.2}$$

to find the following condition,

$$\frac{a_S}{3} A^{-4/3} - \frac{a_C}{6} A^{-1/3} = 0 . \tag{S2.3}$$

The solution is

$$A = \frac{2a_S}{a_C} . \tag{S2.4}$$

From the given numerical values, $A = 50$ (which must be an integer) is concluded.

> ▷ Note: In reality B/A has a maximum for A ranging from $^{56}\mathrm{Fe}$ to $^{62}\mathrm{Ni}$. The discrepancy from the answer in this problem is understood by the approximation of dropping the pairing energy and disregarding a mass difference between the proton and the neutron.

A.1	0.9 pt
$A = 50$	

A.2 Take the differentiation of $B(Z, A-Z)/A$ with respect to Z for a fixed A, which leads to

$$-2a_C \frac{Z^*}{A^{1/3}} - 4a_{\mathrm{sym}} \frac{2Z^* - A}{A} = 0 . \tag{S2.5}$$

By solving this in terms of Z^*, one finds

$$Z^* = \frac{1}{1 + \dfrac{a_C}{4a_{\mathrm{sym}}} A^{2/3}} \cdot \frac{A}{2} . \tag{S2.6}$$

From this expression one can understand that $Z^* \simeq N$ as long as A is small enough, while Z^* becomes far smaller than N for large A. It is obvious from the explicit form that the symmetry energy tends to favor $Z = N$ but the Coulomb interaction tends to favor $Z \to 0$, and the balance between these competing effects determines Z^*. Nuclei with too many neutrons (protons) would go through the β^- decay (the β^+ decay or the electron capture) toward the stable (Z, N).

A.2	0.9 pt
$Z^* = 79$	

A.3 Plugging the binding energy into the given inequality, one sees that the volume terms cancel due to volume conservation. Then, the condition involves only a_S and a_C which are related as

$$a_S\left[A^{2/3} - 2\left(\frac{A}{2}\right)^{2/3}\right] + a_C\left[\frac{Z^2}{A^{1/3}} - 2\frac{(Z/2)^2}{(A/2)^{1/3}}\right] > 0\,. \tag{S2.7}$$

As guided in the problem, the above inequality becomes as simple as

$$\frac{Z^2}{A} > \frac{2^{1/3}-1}{1-2^{-2/3}} \cdot \frac{a_S}{a_C}\,. \tag{S2.8}$$

Therefore, the numerical coefficient turns out to be 0.7.

▷ Note: The physical interpretation of this result may need some explanations. Using the values of a_S and a_C, one finds that such a symmetric fission process is possible for $Z^2/A \gtrsim 18$. For example, lead (Pb) with $Z = 82$ and $A = 208$ is a stable element among several isotopes. Now, one can compute $82^2/208 \approx 32$, which is larger than the threshold 18. The key to resolving this gap is the potential barrier from the deformation. When a heavy nucleus splits into two fragments, the shape and the surface should change from the stable configuration (which is not necessarily spherical due to interaction) and thus the surface energy increases. Although some heavy elements are energetically unstable, the lifetime necessary to overcome the potential barrier can be very large.

A.3	0.7 pt
$C_{\text{fission}} = 7.0 \times 10^{-1}$	

Part B. Neutron star as a gigantic nucleus (1.5 points)

B.1 The expression apart from the parametric dependence on A can be identified as

$$a_{\text{grav}} = \frac{3}{5}\frac{Gm_N^2}{R_0}\,, \tag{S2.9}$$

which is re-expressed in terms of M_P using the given relation to G, leading to

$$a_{\text{grav}} = \frac{3}{5}\frac{\hbar c\, m_N^2}{R_0 M_P^2} = \frac{3}{5}\cdot\frac{197\,\text{fm}\cdot\text{MeV}\times(939\,\text{MeV}/c^2)^2}{1.1\,\text{fm}\times(1.22\times10^{22}\,\text{MeV}/c^2)^2} \simeq 6.4\times10^{-37}\,\text{MeV}\,. \tag{S2.10}$$

Here, M_P is a quantity often called the Planck mass. The gravitational effect is extremely tiny as compared to the typical scale in nuclear physics and this scale difference is manifest for this expression of G with M_P in the MeV unit.

The stability is judged from the condition that the binding energy should be positive, i.e.,

$$B_{\text{total}}(A) = a_V A - a_{\text{sym}} A + a_{\text{grav}} A^{5/3} > 0\,. \tag{S2.11}$$

This inequality can be translated into $A > A_c$ with A_c given by

$$A_c = \left(\frac{a_{\text{sym}} - a_V}{a_{\text{grav}}}\right)^{3/2} \simeq 4.4\times10^{55}\,. \tag{S2.12}$$

▷ Note: One may think that one neutron drip is a process with the least change in the surface area and thus the smallest barrier. This leads to a condition, $B_{\text{total}}(A) > B_{\text{total}}(A-1)$ or approximately $dB_{\text{total}}(A)/dA > 0$, which is satisfied in a window with $B_{\text{total}} < 0$. This condition, $dB_{\text{total}}/dA = 0$, results in smaller A_c but it is nontrivial whether such an unstable initial state could be prepared in the nature. The neutron star is born in the Type-II (core-collapse) supernovae, and a baby star called the proto-neutron star is an energetic state at high temperature. Neutrinos bring heat out from the proto-neutron star within the time scale of $\mathcal{O}(10)$ seconds. What is the possible smallest mass of the neutron star? This is not completely understood partly because the computer simulation of the supernovae is a big challenge even today. Although the neutron star mass can become much smaller than $M_\odot$ theoretically, the simulation and the observation favor the mass $\gtrsim 1.4M_\odot$.

B.1 1.5 pt

$$a_{\text{grav}} = 6 \times 10^{-37}\ \text{MeV}$$

$$A_c = 4 \times 10^{55}$$

Part C. Neutron star in a binary system (6.0 points)

C.1 From the energy conservation, the free-falling system earns the kinetic energy $\frac{1}{2}mv^2$ from the potential energy $mg\Delta h$, and the velocity takes

$$v^2 = 2g\Delta h = 2\Delta\phi\,. \tag{S2.13}$$

The time delay can be derived from the standard arguments. Suppose that Clock-II passes two infinitesimally separate points, z and $z+\Delta z$, in F at time t and $t+\Delta\tau_{\text{II}}$, then the time interval registered by Clock-II is

$$\Delta\tau_{\text{II}} = \frac{\gamma}{c}(c\Delta\tau_{\text{F}} - \beta\Delta z)\,, \tag{S2.14}$$

where the Lorentz transformation is used[1] with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. Because $\Delta z/\Delta\tau_{\text{F}} = v$ and $\Delta\tau_{\text{F}} = \Delta\tau_{\text{I}}$, the above expression is written as

$$\Delta\tau_{\text{II}} = \gamma(1-\beta^2)\Delta\tau_{\text{F}} = \sqrt{1-\beta^2}\,\Delta\tau_{\text{I}}\,. \tag{S2.15}$$

Using the expression of v^2, one finally arrives at

$$\Delta\tau_{\text{II}} = \sqrt{1 - 2\frac{\Delta\phi}{c^2}}\,\Delta\tau_{\text{I}} \simeq \left(1 - \frac{\Delta\phi}{c^2}\right)\Delta\tau_{\text{I}}\,. \tag{S2.16}$$

C.1 1.0 pt

$$\Delta\tau_{\text{II}} = \left(1 - \frac{\Delta\phi}{c^2}\right)\Delta\tau_{\text{I}}$$

C.2 In terms of the effective speed of light, the total time necessary for the light propagation from **N** to **E** is

$$t_{\text{E-N}} = \int_{x_N}^{x_E} \frac{dx}{c_{\text{eff}}(x)}\,. \tag{S2.17}$$

The denominator is expanded in terms of the gravitational potential and the leading-order correction is found to be

$$t_{\text{E-N}} \simeq \frac{1}{c}\int_{x_N}^{x_E} dx\left(1 + \frac{2GM_{\text{WD}}}{c^2\sqrt{x^2+d^2}}\right) = \frac{x_E - x_N}{c} + \Delta t\,, \tag{S2.18}$$

[1] Clock-F is in an inertial frame but Clock-II is not. Using Clock-II′ in another free-falling frame II′ as an inertial reference to Clock-II, the Lorentz transformation is validated for Clock-II′ seen from Clock-F.

where the time delay Δt is identified as

$$\Delta t = \frac{2GM_{\rm WD}}{c^3}\int_{x_N}^{x_E}\frac{dx}{\sqrt{x^2+d^2}} = \frac{GM_{\rm WD}}{c^3}\log\left(\frac{\sqrt{x^2+d^2}+x}{\sqrt{x^2+d^2}-x}\right)\Bigg|_{x=x_N}^{x=x_E}. \tag{S2.19}$$

Inside the logarithm, the following approximations are made:

$$\sqrt{x_N^2+d^2}+x_N \simeq \frac{d^2}{2|x_N|}, \qquad \sqrt{x_N^2+d^2}-x_N \simeq 2|x_N|, \tag{S2.20}$$

and

$$\sqrt{x_E^2+d^2}-x_E \simeq \frac{d^2}{2x_E}, \qquad \sqrt{x_E^2+d^2}+x_E \simeq 2x_E. \tag{S2.21}$$

Then, the simple form of approximated Δt is

$$\Delta t \simeq \frac{GM_{\rm WD}}{c^3}\log\left(\frac{2x_E\cdot 2|x_N|}{d^2/(2x_E)\cdot d^2/(2|x_N|)}\right) = \frac{2GM_{\rm WD}}{c^3}\log\left(\frac{4|x_N|x_E}{d^2}\right). \tag{S2.22}$$

C.2 1.8 pt

$$\Delta t = \frac{2GM_{\rm WD}}{c^3}\log\left(\frac{4|x_N|x_E}{d^2}\right)$$

C.3 Because $|x_N| = L\cos\varepsilon \simeq L$ and $d = L\sin\varepsilon \simeq L\varepsilon$ for $\Delta t_{\rm max}$, the answer of C.2 gives

$$\Delta t_{\rm max} = \frac{2GM_{\rm WD}}{c^3}\log(4x_E/L\varepsilon^2) \tag{S2.23}$$

For $\Delta t_{\rm min}$ the sign of x_N is changed. Although the expression of Δt is intact, the approximation takes a different form as

$$\sqrt{x_N^2+d^2}+x_N \simeq 2x_N, \qquad \sqrt{x_N^2+d^2}-x_N \simeq \frac{d^2}{2x_N}. \tag{S2.24}$$

Then, the approximated form of $\Delta t_{\rm min}$ is

$$\Delta t_{\rm min} \simeq \frac{GM_{\rm WD}}{c^3}\log\left(\frac{2x_E\cdot d^2/(2x_N)}{d^2/(2x_E)\cdot 2x_N}\right) = \frac{2GM_{\rm WD}}{c^3}\log(x_E/L), \tag{S2.25}$$

where $x_N \simeq L$ is used in the last expression. In the difference, $\Delta t_{\rm max} - \Delta t_{\rm min}$, one sees that L and x_E disappear.

C.3 1.8 pt

$$\Delta t_{\rm max} - \Delta t_{\rm min} = \frac{2GM_{\rm WD}}{c^3}\log(4/\varepsilon^2)$$

C.4 Using the expansion, $\cos\varepsilon \simeq 1 - \frac{1}{2}\varepsilon^2$, one can evaluate

$$\varepsilon^2 \simeq 2\times(1-\cos\varepsilon) = 0.00022. \tag{S2.26}$$

From the graph the difference in time delays is roughly read out as

$$\Delta t_{\rm max} - \Delta t_{\rm min} \approx 50\,\mu\text{s} \tag{S2.27}$$

From these numerical values, $M_{\rm WD}$ is solved as

$$M_{\rm WD} = M_\odot\left(\frac{2GM_\odot}{c^3}\right)^{-1}\frac{\Delta t_{\rm max}-\Delta t_{\rm min}}{\log(4/\varepsilon^2)} \simeq \frac{50\,\mu\text{s}}{10\,\mu\text{s}\ \log(4/0.00022)}M_\odot \simeq 0.5\,M_\odot. \tag{S2.28}$$

▷ Note: The data in this problem roughly correspond to PSR J1614-2230 [see P.B. Demorest *et al.*, Nature 467, 1080-1083 (2010)]. From the Shapiro delay measurement, the White Dwarf mass was estimated as $0.500 \pm 0.006 M_\odot$. With the Keplerian orbital parameters in the binary system, the neutron star mass was considered to be $1.97 \pm 0.04 M_\odot$, which was the heaviest neutron star at that time. Since then, several candidates for heavier neutron stars have been found.

C.4 0.8 pt

$M_{\mathrm{WD}}/M_\odot = 0.5$

C.5 For the circular orbit with the radius R, the equation of motion is

$$mR\omega^2 = G\frac{mM}{R^2}\,, \tag{S2.29}$$

if M is sufficiently large. From this it is easy to see

$$R^3\omega^2 = GM = (\text{const.}) \tag{S2.30}$$

This is nothing but Kepler's third law and the relation holds for more general elliptical orbits around the center of mass.

C.5 0.4 pt

$p = -\frac{3}{2}$

C.6 The sum of the kinetic energy and the potential energy is

$$E = \frac{1}{2}mR^2\omega^2 - G\frac{mM}{R}\,. \tag{S2.31}$$

From the equation of motion this is rewritten as

$$E = -\frac{1}{2}G\frac{mM}{R}\,. \tag{S2.32}$$

Therefore, when E decreases due to gravitational wave emission, R should decrease. Then, ω should increase. Since the amplitude is proportional to $R^2\omega^2 \propto R^{-1}$, it should increase. In summary, both the frequency and the amplitude should increase as time goes on, as illustrated in (b).

C.6 0.2 pt

The most appropriate profile is (b).

Theory Problem 3: Water and Objects (10 points)

Part A. Merger of water drops (2.0 pt)

A.1 The surface energy per drop before the merger is

$$E = 4\pi a^2 \gamma. \tag{S3.1}$$

Therefore, the surface energy difference becomes

$$\Delta E = 4\pi\left(2 - 2^{2/3}\right) a^2 \gamma. \tag{S3.2}$$

The transfer of surface energy to kinetic energy is represented by

$$Mv^2/2 = k\Delta E, \tag{S3.3}$$

where $k = 0.06$ and $M = 4\pi a^3 \rho/3 \times 2 = 8\pi a^3\rho/3$ is the mass of the drop after the merger. The numerical computation gives

$$v = \sqrt{\frac{2k\Delta E}{M}} = \sqrt{3\left(2 - 2^{2/3}\right)\frac{k\gamma}{\rho a}} = \sqrt{3\left(2-2^{2/3}\right) \times \frac{0.06 \times \left(7.27 \times 10^{-2}\right)}{\left(1.0 \times 10^3\right) \times \left(100 \times 10^{-6}\right)}} = 0.23\not{2}\ \mathrm{m/s}. \tag{S3.4}$$

▷ Note: We point out an interesting phenomenon related to this question. On a superhydrophobic surface, when small droplets merge, they release surface energy, causing the surface area to shrink. This energy release propels the merged droplet to jump up. This phenomenon mirrors the natural mechanism seen in cicadas. Cicadas' wings, which possess superhydrophobic surfaces, facilitate the removal of water droplets upon coalescence. This process serves as a natural self-cleaning system, converting surface energy to kinetic energy, as reported in the following paper: Wisdom *et. al.*, Proc. Natl. Acad. Sci. USA 110, 7992–7997 (2013).

A.1	2.0 pt
$v = 0.23\ \mathrm{m/s}$	

Part B. A vertically placed board (4.5 pt)

B.1

Consider a vertical upright column-shaped water block as shown in the hatched area of Fig. S3-1. The vertical force balance with respect to this block per unit area leads to $P + \rho g z = P_0$.

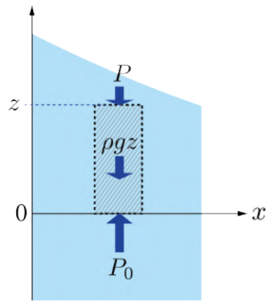


Fig. S3-1

B.1	0.6 pt
$P = P_0 - \rho g z$	

B.2

Because the atmospheric pressure P_0 exerts no net horizontal force on the water block, we have

$$f_x = \int_{z_2}^{z_1} (-\rho g z) dz = \frac{1}{2}\rho g\left(z_2^2 - z_1^2\right). \tag{S3.5}$$

This force acts in the leftward direction.

▷ Note: The reason why P_0 exerts no net horizontal force is understood as follows. Consider a small area (infinitesimally divided piece) near the surface, which is regarded as a right-angled triangle (see Fig. S3-2).

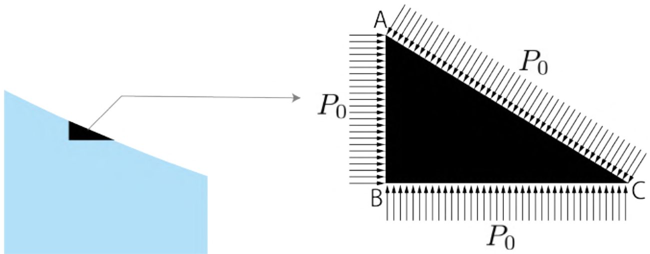


Fig. S3-2

The horizontal component of the combined force exerted on the right-angled triangle ABC per unit length along the y-axis by atmospheric pressure is

$$P_0 \times \overline{\mathrm{AB}} - P_0 \times \overline{\mathrm{AC}} \times \frac{\overline{\mathrm{AB}}}{\overline{\mathrm{AC}}} = 0$$

Integrating infinitesimal pieces over a finite domain yields a finite volume of water, while the net force remains zero.

B.2 0.8 pt

$$f_x = \frac{1}{2}\rho g \left(z_2^2 - z_1^2\right)$$

B.3

The horizontal component of the surface tension acting on the water block is $\gamma \cos\theta_2 - \gamma \cos\theta_1$. Thus, the horizontal force balance is expressed as

$$f_x + \gamma \cos\theta_2 - \gamma \cos\theta_1 = 0. \tag{S3.6}$$

B.3 0.8 pt

$$f_x = \gamma \cos\theta_1 - \gamma \cos\theta_2$$

B.4

From the results of B.2 and B.3, we have

$$\frac{1}{2}\rho g z_1^2 + \gamma \cos\theta_1 = \frac{1}{2}\rho g z_2^2 + \gamma \cos\theta_2. \tag{S3.7}$$

Since this equation holds at an arbitrary point (x, z) on the water surface, we conclude

$$\frac{1}{2}\rho g z^2 + \gamma \cos\theta = \text{constant}, \tag{S3.8}$$

which is written as

$$\frac{1}{2}\left(\frac{z}{\ell}\right)^a + \cos\theta(x) = \text{constant}, \tag{S3.9}$$

with $a = 2$ and $\ell = \sqrt{\frac{\gamma}{\rho g}}$.

▷ Note: The equaiton (S3.9) is a kind of conservation law. The constant ℓ is called the capillary length.

B.4 0.8 pt

$$a = 2, \quad \ell = \sqrt{\frac{\gamma}{\rho g}}$$

B.5 The derivative of the water surface coordinate z, denoted by z', is associated with the angle of inclination θ, given by the equation: $z' = \tan\theta$. This relation yields

$$\cos\theta = \frac{1}{\sqrt{1 + (z')^2}}, \tag{S3.10}$$

which leads to

$$\cos\theta \simeq 1 - \frac{1}{2}(z')^2. \tag{S3.11}$$

Plugging this into Eq. (S3.9), we obtain

$$\frac{z^2}{\ell^2} - z'^2 = \text{const.} \tag{S3.12}$$

Taking the derivative of both sides with respect to x, we have

$$z'' = \frac{z}{\ell^2}, \tag{S3.13}$$

which is the differential equation that determines the water surface form.

Its general solution is

$$z = Ae^{x/\ell} + Be^{-x/\ell}. \tag{S3.14}$$

The boundary condition, $z(\infty) = 0$, leads to $A = 0$.

The boundary condition, $z'(0) = \tan\theta_0$, leads to $B = -\ell\tan\theta_0$.

B.5 1.5 pt

$$z(x) = -\ell\tan\theta_0 e^{-x/\ell}$$

Part C. Interaction between two rods (3.5 pt)

C.1 The horizontal component of the force due to the pressure is

$$\int_{z_\mathrm{a}}^{z_\mathrm{b}} (\rho g z)\, dz = \frac{1}{2}\rho g \left(z_\mathrm{b}^2 - z_\mathrm{a}^2\right) \tag{S3.15}$$

Let z_bottom be the z-coordinate at the bottom of the rod. Then, we have

$$F_x = \int_{z_\mathrm{bottom}}^{z_\mathrm{a}} (-\rho g z) dz + \left(- \int_{z_\mathrm{bottom}}^{z_\mathrm{b}} (-\rho g z) dz \right) = \int_{z_\mathrm{a}}^{z_\mathrm{b}} (\rho g z) dz \tag{S3.16}$$

▷ Note: The fact that the contribution due to the pressure does not depend on the shape of the cross-section can be demonstrated as follows. The pressure at the point s on the contour C along the cross-sectional boundary is

$$-P\hat{n}ds = (-P_0 + \rho g z)\hat{n}ds. \tag{S3.17}$$

Let $\hat{x}$ be the unit vector pointing the positive x-direction and noting $\hat{x} \cdot \hat{n} ds = dz$ (see Fig. S3-3), we obtain its horizontal component as

$$-P\hat{n} \cdot \hat{x} ds = -P_0 dz + \rho g z dz. \tag{S3.18}$$

Integrating along the contour C.[1] We obtain

$$\int_{z_\mathrm{a}}^{z_\mathrm{b}} (\rho g z) dz = \frac{1}{2}\rho g \left(z_\mathrm{b}^2 - z_\mathrm{a}^2\right) \tag{S3.19}$$

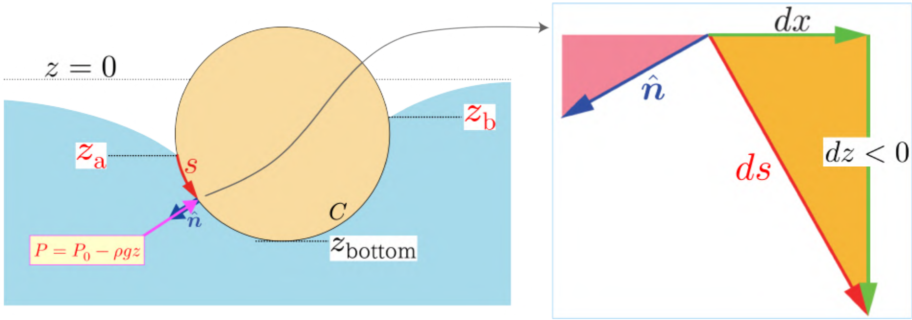


Fig. S3-3

C.1 1.0 pt

$$F_x = \frac{1}{2}\rho g \left(z_\mathrm{b}^2 - z_\mathrm{a}^2\right) + \gamma \left(\cos\theta_\mathrm{b} - \cos\theta_\mathrm{a}\right)$$

C.2 By applying the boundary conditions to Eq. (S3.8), we obtain

$$\underbrace{\frac{1}{2}\rho g z_\mathrm{a}^2 + \gamma \cos\theta_\mathrm{a}}_{x=x_\mathrm{a}} = \underbrace{\frac{1}{2}\rho g z_0^2 + \gamma}_{x=0} \tag{S3.20}$$

$$\underbrace{\frac{1}{2}\rho g z_\mathrm{b}^2 + \gamma \cos\theta_\mathrm{b}}_{x=x_\mathrm{b}} = \underbrace{\gamma}_{x\to\infty} \tag{S3.21}$$

Then, $F_x = -\frac{1}{2}\rho g z_0^2$ is obtained by subtracting (S3.20) from (S3.21).

▷ Note: The physical background of this problem is as follows. When a single rod is placed on the water surface, the shape of the water surface on both sides of the rod remains the same. In other words, the rod is placed in an environment that exhibits the left-right symmetry. Then, there is no force acting on the rod. On the other hand, when two rods are placed on the water surface, the left-right symmetry of the water surface is broken from the perspective of each rod. As a result, an attractive force is generated.

The displacement of the water surface at the midpoint between the two rods differs from that of a horizontal water surface. This deviation is represented by z_0. That is to say, z_0 plays the role of a symmetry-breaking parameter (see Fig. S3-4).

The fact that the attractive force between the two rods is determined solely by this parameter suggests that the symmetry breaking directly becomes the origin of the force. This corresponds to the fundamental principle in physics that relates symmetry breaking to force generation.

[1] This integral is symbolically written as $\oint_C (-P\hat{n} \cdot \hat{x} ds)$.

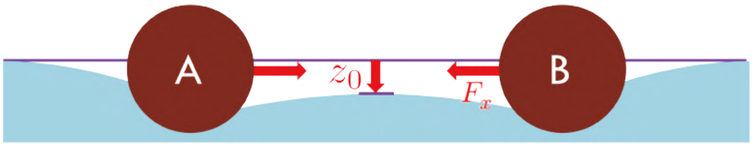


Fig. S3-4

C.2 1.5 pt

$$F_x = -\frac{1}{2}\rho g z_0^2$$

C.3 The general solution for the water surface height is given by the equation

$$z(x) = Ae^{x/\ell} + Be^{-x/\ell}. \tag{S3.22}$$

By considering the left-right symmetry, we find

$$A = B. \tag{S3.23}$$

Applying the boundary condition $z(0) = z_0$, we obtain

$$A + B = z_0. \tag{S3.24}$$

We thus have

$$A = z_0/2, \quad B = z_0/2. \tag{S3.25}$$

C.3 1.0 pt

$$z_0 = \frac{2z_a}{e^{x_a/\ell} + e^{-x_a/\ell}}$$