Solution

Water Hammer

Part A. *Excess Pressure and Propagation of Pressure wave*

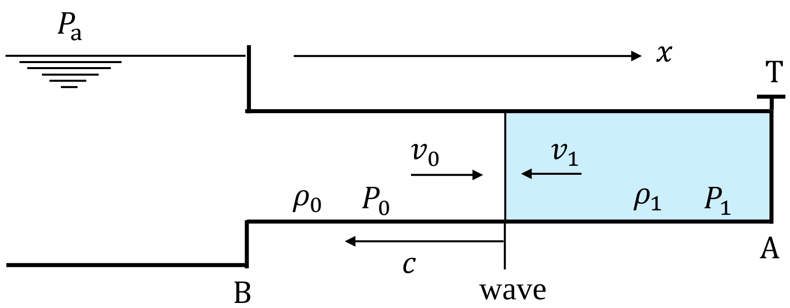


Fig. S1. Pressure wave (shaded) with speed c

A.1 (1.6 pt) Excess pressure and speed of propagation of the pressure wave

When the valve opening is suddenly blocked, fluid pressure at the valve jumps from P_0 to $P_1 = P_0 + \Delta P_s$, thus sending a pressure wave traveling upstream (to the left) with speed c and amplitude ΔP_s. Taking positive x direction as pointing to the right, the velocity of fluid particles next to the valve changes from v_0 to v_1 $(v_1 \leq 0)$. Thus the velocity change is $\Delta v = v_1 - v_0$.

In a frame moving to left (along $-x$ direction) with speed c, i.e., riding on the wave (see Fig. S1), velocity of fluid in the pressure wave is $c + v_1$, while that of the incoming fluid in the steady flow ahead of the wave is $c + v_0$. Let ρ_1 be the density of fluid in the pressure wave. From conservation of mass, i.e., equation of continuity, we have

$$\rho_0(c + v_0) = \rho_1(c + v_1) \qquad (a1)$$

or, by letting $\Delta\rho \equiv \rho_1 - \rho_0$,

$$\frac{\Delta\rho}{\rho_1} = 1 - \frac{\rho_0}{\rho_1} = \frac{v_0 - v_1}{c + v_0} = \frac{-\Delta v}{c + v_0} \qquad (a2)$$

Moreover, impulse imparted to the fluid must equal its momentum change. Thus, in a short time interval τ after the valve is closed, we must have

$$\rho_0(c + v_0)\tau[(c + v_1) - (c + v_0)] = -\tau\Delta P = (P_0 - P_1)\tau \qquad (a3)$$

or

$$\Delta P_s = -\rho_0 c\left(1 + \frac{v_0}{c}\right)(v_1 - v_0) = -\rho_0 c\left(1 + \frac{v_0}{c}\right)\Delta v \quad \Rightarrow \quad \alpha = -\left(1 + \frac{v_0}{c}\right) \qquad (a4)$$

If $v_0/c \ll 1$, we have

$$\Delta P_s = -\rho_0 c \Delta v \qquad (a5)$$

Note that the *negative* sign in Eqs. (a4) and (a5) follows from the fact that the direction of propagation is opposite to the positive direction for x axis (and velocity). Otherwise the sign should be *positive*. Note also that for a compressional wave

($\Delta P_s > 0$), the velocity imparted to the fluid particle is in the direction of propagation, while for an extensional wave ($\Delta P_s < 0$), the velocity imparted is in the opposite direction of propagation.

Eqs. (a2) and (a4) can be combined to give

$$\Delta P_s = \rho_0 c^2 \left(1 + \frac{v_0}{c}\right)^2 \frac{\Delta\rho}{\rho_1} \tag{a6}$$

From the definition of the bulk modulus B, which is assumed to be constant, it follows

$$\Delta P_s = B\frac{V_0 - V_1}{V_0} = B\frac{1/\rho_0 - 1/\rho_1}{1/\rho_0} = B\frac{\Delta\rho}{\rho_1} \tag{a7}$$

From Eqs. (a6) and (a7), we obtain

$$\rho_0 c^2 \left(1 + \frac{v_0}{c}\right)^2 = B \tag{a8}$$

Thus

$$c = \sqrt{\frac{B}{\rho_0}} - v_0 \qquad \Rightarrow \quad \gamma = 1 \qquad \beta = -v_0 \tag{a9}$$

However, if in the definition of bulk modulus one uses the fractional change of density $\Delta\rho/\rho_0$ instead of $-\Delta V/V_0$, the result is then $\gamma = 1 + \Delta P_s/B$.* Either result is considered valid.

If $v_0/c \ll 1$, we have

$$c = \sqrt{\frac{B}{\rho_0}} \tag{a10}$$

*The result (a7) is pointed out by Dr. Jaan Kalda.

A.2 (0.6 pt) Values of c and ΔP_s for water flow

Ans:

From Eqs. (a5) and (a10), we have

$$c = \sqrt{B/\rho_0}$$
$$\Delta P_s = \rho_0 c v_0 = v_0\sqrt{\rho_0 B}$$

Putting in the given values $v_0 = 4.0\ \mathrm{m/s}$, $v_1 = 0$, $\rho_0 = 1.0 \times 10^3\ \mathrm{kg/m^3}$, and $B = 2.2 \times 10^9$ Pa, we have

$$c = \sqrt{B/\rho_0} = 1.5 \times 10^3\ \mathrm{m/s} \tag{b1}$$
$$\Delta P_s = v_0\sqrt{\rho_0 B} = 5.9\ \mathrm{MPa} \tag{b2}$$

so that ΔP_s is nearly 59 times the standard pressure.

Note that $v_0/c \sim 10^{-3}$ so that the use of approximate formulas (a5) and (a10) is justified when solving tasks in this problem.

Part B. *A Model for the Flow-Control Valve*

(B.1) (1.0 pt) Excess pressure at valve inlet

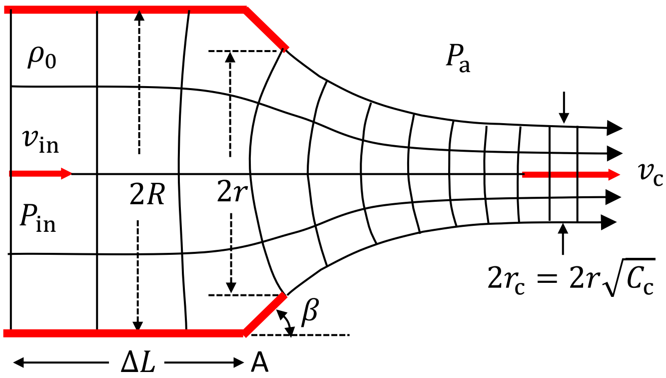


Fig. 2. Valve dimensions and contraction of jet.

Ans:

The model assumes the fluid to be incompressible. Neglecting effects of gravity, Bernoulli's principle gives us

$$\frac{1}{2}\rho_0 v_{in}^2 + P_{in} = \frac{1}{2}\rho_0 v_c^2 + P_a \tag{c1}$$

Equation of continuity and definition of contraction coefficient imply that

$$\pi R^2 v_{in} = \pi r_c^2 v_c = \pi r^2 C_c v_c$$

Therefore

$$v_c = \frac{1}{C_c}\left(\frac{R}{r}\right)^2 v_{in} \tag{c2}$$

From Eqs. (c1) and (c2), we obtain

$$\Delta P_{in} = P_{in} - P_a = \frac{1}{2}\rho_0 v_{in}^2 \left[\frac{1}{C_c^2}\left(\frac{R}{r}\right)^4 - 1\right] = \frac{k}{2}\rho_0 v_{in}^2 \tag{c3}$$

This may be cast into a form involving only dimensionless variables:

$$\frac{\Delta P_{in}}{\rho_0 c^2} = \frac{1}{2}\left(\frac{v_{in}}{c}\right)^2 \left[\frac{1}{C_c^2}\left(\frac{R}{r}\right)^4 - 1\right] = \frac{k}{2}\left(\frac{v_{in}}{c}\right)^2 \tag{c4}$$

where

$$k = \left[\frac{1}{C_c^2}\left(\frac{R}{r}\right)^4 - 1\right] \tag{c5}$$

Thus we see from eq. (c4) that ΔP_{in} is a quadratic function of v_{in}.

Part C. *Water-Hammer Effect due to Fast Closure of Flow-Control Valve*

(C.1) (0.6 pt) Pressure P_0 and velocity v_0 when the valve is fully open

Ans:

According to Bernoulli's theorem and the definition of P_h, we have

$$\frac{1}{2}\rho_0 v_0^2 + P_0 = \frac{1}{2}\rho_0 v_c^2 + P_a = 0 + P_a + \rho_0 gh = P_h \qquad \text{(d1)}$$

From the second equality in the preceding equation, it follows

$$v_c = \sqrt{2gh}$$

Furthermore, from continuity equation and $C_c(r = R) = 1.0$, we have

$$\pi R^2 v_0 = \pi (C_c R)^2 v_c = \pi R^2 v_c \Rightarrow v_0 = v_c = \sqrt{2gh} \qquad \text{(d2)}$$

Therefore

$$P_0 = P_a = P_h - \rho_0 gh \qquad \text{(d3)}$$

(C.2) (1.2 pt) Pressure $P(t)$ and flow velocity $v(t)$ just before $t = \frac{\tau}{2} = \frac{L}{c}$ and $t = \tau$

Ans:

When the valve is open, the flow in the pipe is steady with velocity v_0 and pressure P_0. The sudden closure of the valve causes an excess pressure ΔP_s on the fluid element next to the valve, causing it to stop with velocity $v_1 = 0$. The velocity change is thus $\Delta v = v_1 - v_0 = -v_0$. Thus, according to Eq. (a5), the excess pressure on the fluid is given by

$$\Delta P_s = -\rho_0 c \Delta v = \rho_0 c v_0 \qquad \text{(e1)}$$

At time $t = \tau/2 = L/c$, the pressure wave reaches the reservoir. The velocity of fluid in the length of the pipe has all changed to $v(\tau/2) = v_1 = v_0 + \Delta v = 0$ and the fluid pressure is $P(\tau/2) = P_1 = P_0 + \Delta P_s = P_0 + \rho_0 c v_0$.

At the reservoir end of the pipe, fluid pressure reduces to the constant hydrostatic pressure $P_h = P_0 + \rho_0 gh$. Equivalently, we may say that the reservoir acts as a free end for the pressure wave and, in reducing its excess pressure to P_h, causes a compression wave to be reflected as an expansion wave. Relative to the hydrostatic pressure P_h, the amplitude of the incoming pressure wave is $\Delta P_{1r} = P_1 - P_h$, hence the reflected expansion wave will have an amplitude $\Delta P_1' = -\Delta P_{1r}$ and we have

$$\Delta P_1' = -\Delta P_{1r} = P_h - P_1 = (P_0 + \rho_0 gh) - (P_0 + \rho_0 c v_0) = -\rho_0 c (v_0 - gh/c) \qquad \text{(e2)}$$

(Here we allow the pressure amplitude to have both signs with negative amplitude signifying an expansion wave.) This will cause the fluid at the reservoir end of the pipe to suffer a velocity change (keeping in mind that the direction of propagation is now the same as the $+x$ axis)

$$\Delta v_{1r} = +\Delta P_1'/(\rho_0 c) = -(v_0 - gh/c)$$

Consequently, its velocity changes to

$$v_{1r} = v_1 + \Delta v_{1r} = 0 - \left(v_0 - \frac{gh}{c}\right) \qquad \text{(e3)}$$

Ahead of the front of the reflected wave, conditions are unchanged and the particle velocity is still $v_1 = 0$ and the fluid pressure is still $P_1 = P_0 + \Delta P_s$, but behind the wave front the particle velocity now becomes $v_{1r} = -(v_0 - gh/c)$ and the pressure becomes

$$P_1 + \Delta P_1' = (P_0 + \rho_0 c v_0) - \rho_0 c\left(v_0 - \frac{gh}{c}\right) = P_0 + \rho_0 gh \qquad \text{(e4)}$$

Therefore, just moment before $t = \tau = 2L/c$ when the front of the reflected wave reaches the valve, the fluid in the whole length of the pipe will be under the pressure $P(\tau) = P_0 + \rho_0 gh = P_h$ as given in Eq. (e4) , and all fluid particles in the pipe will move, as given in Eq. (e3), with velocity $v(\tau) = v_{1r} = -v_0 + gh/c$, i.e., the fluid in the pipe is expanding and flowing toward the reservoir.

Part D. *Water-Hammer Effect due to Slow Closure of Flow-Control Valve*

(D.1) (3.0 pt) Recursion relations for ΔP_n and v_n

Ans:

Enforcing the approximation $P_h = P_0 + \rho_0 gh \approx P_0$ is equivalent to putting $h = 0$ in all of the results obtained in task (e).

(1) Partial closing $n = 1$

At the valve, immediately after partial closing $n = 1$, fluid pressure jumps from P_0 to P_1, causing flow velocity to change from v_0 to v_1. The pressure and velocity changes are related by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_1 - P_0) = -(v_1 - v_0) \qquad \text{(f1)}$$

Just before reflection by the reservoir, the fluid in the entire pipe has pressure P_1 and velocity v_1. After reflection by the reservoir, i.e., a free end, and before the start of valve closure $n = 2$, the fluid in the entire pipe has pressure (Eq. (e4) with $h = 0$)

$$P_1 - (P_1 - P_0) = P_0$$

and velocity

$$v_1' = v_1 + \frac{-(P_1 - P_0)}{\rho_0 c} = v_1 + (v_1 - v_0)$$

(2) Partial closing $n = 2$

Immediately after partial closing $n = 2$, valve pressure changes from P_0 to P_2, causing flow velocity to change from v_1' to v_2. The pressure and velocity changes are given by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_2 - P_0) = -(v_2 - v_1') = -v_2 + v_1 + (v_1 - v_0) \qquad \text{(f2)}$$

Using Eq. (f1), we may rewrite the preceding equation as

$$\frac{1}{\rho_0 c}(P_2 - P_0) = -(v_2 - v_1) - \frac{1}{\rho_0 c}(P_1 - P_0) \qquad \text{(f3)}$$

Just before reflection by the reservoir, the fluid in the entire pipe has pressure P_2 and velocity v_2. After reflection by the reservoir and before valve closure $n = 3$, the fluid in the entire pipe has pressure

$$P_2 - (P_2 - P_0) = P_0$$

and velocity

$$v_2' = v_2 + (v_2 - v_1')$$

(3) Partial closing $n = 3$

Immediately after partial closing $n = 3$, valve pressure changes from P_0 to P_3, causing flow velocity to change from v_2' to v_3. The pressure and velocity changes are given by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_3 - P_0) = -(v_3 - v_2') = -v_3 + v_2 + (v_2 - v_1') \tag{f4}$$

Using Eq. (f2), we may rewrite the preceding equation as

$$\frac{1}{\rho_0 c}(P_3 - P_0) = -(v_3 - v_2) - \frac{1}{\rho_0 c}(P_2 - P_0) \tag{f5}$$

Just before reflection by the reservoir, the fluid in the entire pipe has pressure P_3 and velocity v_3. After reflection by the reservoir and before valve closure $n = 4$, the fluid in the entire pipe has pressure

$$P_3 - (P_3 - P_0) = P_0$$

and velocity

$$v_3' = v_3 + (v_3 - v_2')$$

(4) Partial closing $n = 4$

When the valve is fully shut at valve closing $n = 4$, the valve becomes a fixed end, so the fluid velocity at the valve changes from v_3' to $v_4 = 0$. The pressure P_4 at the valve is then given by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_4 - P_0) = -(v_4 - v_3') = -v_4 + v_3 - \frac{1}{\rho_0 c}(P_3 - P_0) \tag{f6}$$

Finally, if we take note of the fact that $\Delta P_0 = 0$ and $v_4 = 0$, then all equations obtained above relating excess pressures and velocity changes after valve closings all have the same form:

$$\frac{\Delta P_n}{\rho_0 c} = -(v_n - v_{n-1}) - \frac{\Delta P_{n-1}}{\rho_0 c} \qquad (n = 1,2,3,4) \tag{f7}$$

To solve for $\Delta P_n = P_n - P_0$, we note that, from Eqs. (c3) and (c5), we have another relation between ΔP_n and v_n:

$$\Delta P_n = \frac{1}{2} k_n \rho_0 {v_n}^2 \qquad (n = 1,2,3) \tag{f8}$$

where C_n represents C_c for $r = r_n$ and

$$k_n = \left[\frac{1}{C_n^2}\left(\frac{R}{r_n}\right)^4 - 1\right] \qquad (n = 1,2,3) \tag{f9}$$

Combining Eqs. (f7) and (f8), we have a quadratic equation for v_n:

$$\frac{1}{2} k_n \left(\frac{v_n}{c}\right)^2 + \frac{v_n}{c} + \left(\frac{\Delta P_{n-1}}{\rho_0 c^2} - \frac{v_{n-1}}{c}\right) = 0 \qquad (n = 1,2,3) \tag{f10}$$

which can be solved readily using the formula

$$\frac{v_n}{c} = \frac{-1 + \sqrt{1 + 2k_n\left(\frac{v_{n-1}}{c} - \frac{\Delta P_{n-1}}{\rho c^2}\right)}}{k_n} \qquad (n = 1,2,3) \tag{f11}$$

If both $\Delta P_{n-1}/(\rho c^2)$ and (v_{n-1}/c) are known, Eq. (f11) may be used to compute v_n/c and then find $\Delta P_n/(\rho c^2)$ by using Eq. (f8). Therefore, Eq. (f7) may be solved iteratively starting with $n = 1$ until $n = 3$. For $n = 4$, we know $v_n = 0$, so Eq. (f7) may be used directly to find ΔP_n.

Note that, from Eq. (f8), ΔP_{n-1} is a quadratic function of v_{n-1}, so that if v_{n-1} is known, then v_n may be computed using Eq. (f11) and then ΔP_n may again be computed using Eq. (f8).

(D.2) (2.0 pt) Estimating ΔP_n and $\rho_0 c v_n$ by graphical method

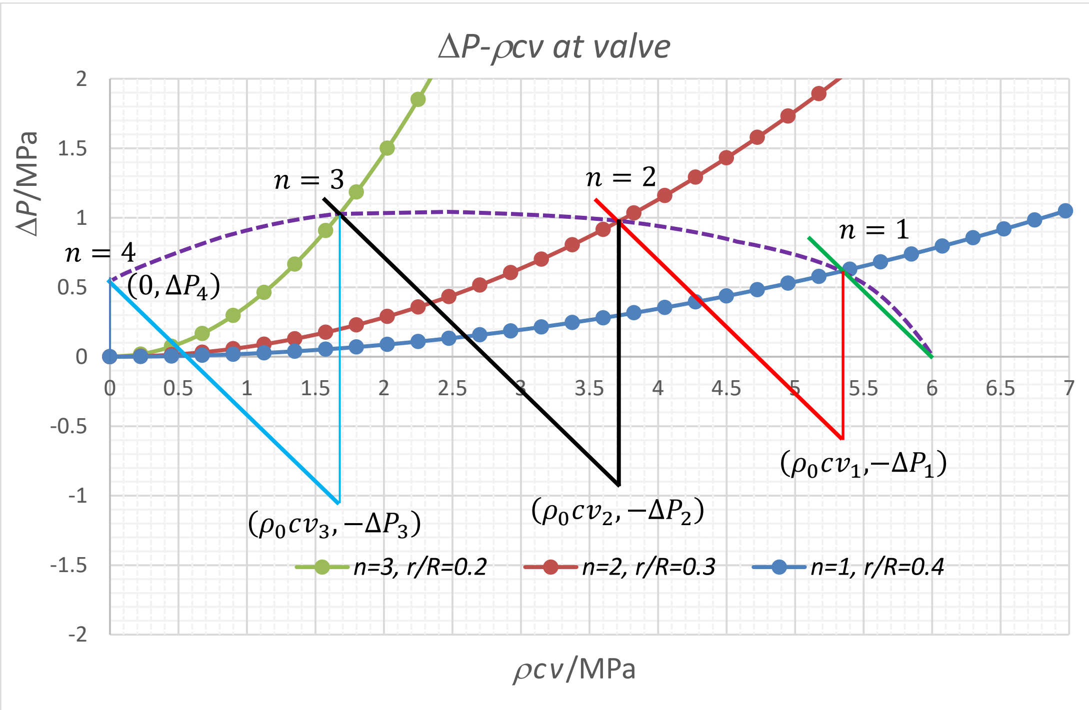
Ans:

To solve Eqs. (f7) and (f8) using graphical method, we rewrite them as follows:

$$\Delta P_n = -(\rho_0 c v_n - \rho_0 c v_{n-1}) - \Delta P_{n-1} \qquad (n = 1,2,3,4) \qquad \text{(g1)}$$

$$\Delta P_n = \frac{k_j}{2\rho_0 c^2}(\rho_0 c v_n)^2 \qquad (n = 1,2,3,4) \qquad \text{(g2)}$$

In a plot of ΔP vs. $\rho_0 c v$, Eq. (g1) and Eq. (g2) correspond to a line passing through the point $(\rho_0 c v_{n-1}, -\Delta P_{n-1})$ with slope -1 and a parabola passing through the origin, respectively. Thus one may readily obtain the solutions for each step of valve closing by locating their points of intersection, starting with $n = 1$. The result is shown in the following graph.



Excess Pressures and particle velocities at the valve for slow closing							
n	r_n/R	C_n	k_n	$v_n/(\mathrm{m/s})$	$\rho_0 c v_n/\mathrm{MPa}$	$\Delta P_n/(\mathrm{MPa})$	$\Delta P_n/(\rho_0 c v_0)$
0	1.00	1.00	0.0	4.0	6.0	0.0	0.0
1	0.40	0.631	97.1	3.6	5.8	0.62	10 %
2	0.30	0.622	318.	2.5	3.8	1.0	17 %
3	0.20	0.616	1646.	1.1	1.7	1.1	18 %

4	0.00			0.0	0.0	0.64	11 %

$\rho_0 c = 1.50 \times 10^6$ kg m^{-2} s^{-1} $\qquad v_0 = 4.0$ m/s

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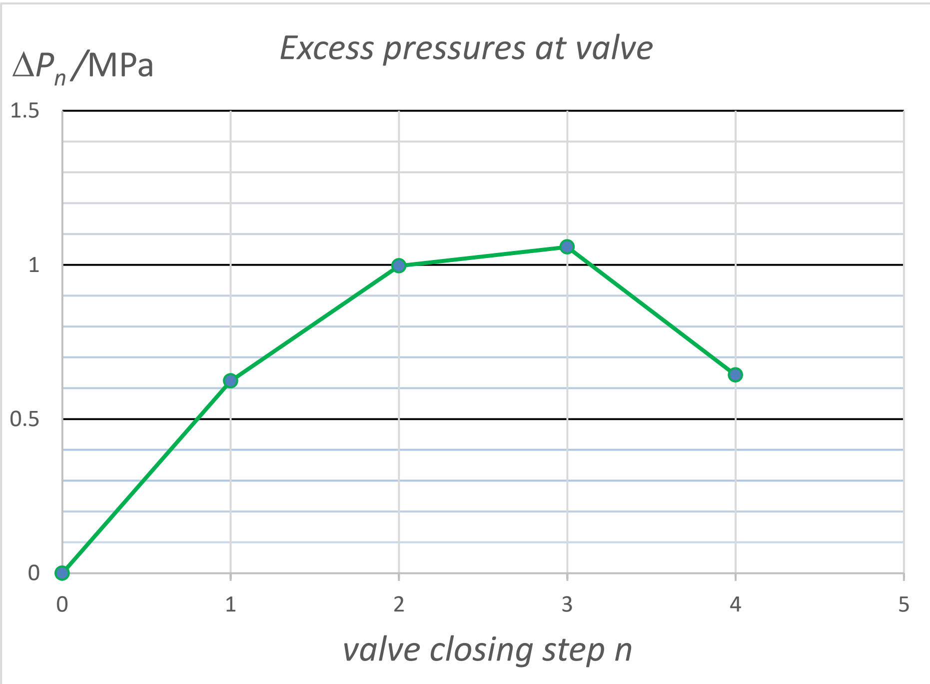
Appendix

(The following table and graph are for reference only, not part of the task.)

For $v_0 = 4.0$ m/s, $c = 1.5 \times 10^3$ m/s, and $\rho = 1.0 \times 10^3$ kg/m^3, the results for v_n and ΔP_n are shown in the following table and graph. They are computed according to equations given in task (f). Note that for a sudden full closure of the valve, we have $\Delta P_{\text{sudden}} = \rho c v_0 = 6.0$ MPa.

Excess Pressures and particle velocities at the valve for slow closing							
n	r_n/R	C_n	k_n	$v_n/(\text{m/s})$	$\rho c v_n/\text{MPa}$	$\Delta P_n/(\text{MPa})$	$\Delta P_n/(\rho c v_0)$
0	1.00	1.00	0.0	4.0	6.0	0.0	0.0
1	0.40	0.631	97.1	3.58	5.37	0.624	10 %
2	0.30	0.622	318.	2.50	3.75	0.997	17 %
3	0.20	0.616	1646.	1.13	1.695	1.06	18 %
4	0.00			0.0	0.0	0.643	11 %



レイトレーシング (光線追跡) ともつれた光の生成 10.0 点

答えとその解答例

Part A. 等方的な誘電体における光の伝搬（1.0 点）

A.1　0.4 点

答え：

$$\frac{1}{\sqrt{\mu_0\epsilon}}$$

解答例：
$\vec{k}\times\vec{E}=\omega\vec{B}=\omega\mu_0\vec{H}$，$\vec{k}\times\vec{H}=-\omega\vec{D}$，より，$\vec{k}\times(\vec{k}\times\vec{E})=-\omega^2\mu_0\vec{D}$ を得る。与えられた等式

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

より，$\vec{k}\times(\vec{k}\times\vec{E})=\vec{k}(\vec{k}\cdot\vec{E})-k^2\vec{E}$ を得る。
$\vec{D}\cdot\vec{k}=0$，$\vec{D}=\epsilon\vec{E}$，より $\vec{k}\times(\vec{k}\times\vec{E})=-k^2\vec{E}$，となり，
$\vec{k}\times(\vec{k}\times\vec{E})=-\omega^2\mu_0\vec{D}$ は，$-k^2\vec{E}=-\omega^2\mu_0\epsilon\vec{E}$，となる。
ここで位相速度は $\dfrac{\mathrm{d}(\vec{k}\cdot\vec{r}-\omega t)}{\mathrm{d}t}$ より求められるので，$\vec{v}_p=\dfrac{\mathrm{d}\vec{r}}{\mathrm{d}t}=\dfrac{\omega}{k}\hat{k}$，となる。明らかに　$\dfrac{\omega}{k}=\dfrac{1}{\sqrt{\mu_0\epsilon}}$ である。よって　$v_p=\dfrac{1}{\sqrt{\mu_0\epsilon}}$ である。

A.2　0.2 点

答え：

$$c\sqrt{\mu_0\epsilon}$$

解答例：
$v_p=\dfrac{1}{\sqrt{\mu_0\epsilon}}=\dfrac{c}{n}$，より $n=c\sqrt{\mu_0\epsilon}$。

A.3　0.4 点

答え：

$$\hat{k},\quad v_r = v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$$

解答例：

光の速さを求めるために，まず，エネルギーの流れの方向はポインティングベクトル $\vec{S} = \vec{E} \times \vec{H}$ で与えられ，これは $\vec{k}$ の方向と一致している。

電磁エネルギー密度は，$u_e = \frac{1}{2}\vec{E}\cdot\vec{D}$，および $u_m = \frac{1}{2}\vec{B}\cdot\vec{H}$，により $u = u_e + u_m$ である。

さて，$\vec{k} \times \vec{H} = -\omega\vec{D}$ から，$\vec{D} = -\frac{1}{v_p}\hat{k} \times \vec{H}$ を得る。

したがって，$u_e = -\frac{1}{2v_p}\vec{E}\cdot\hat{k}\times\vec{H} = \frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$。

また，同様に $\vec{k} \times \vec{E} = \omega\vec{B}$，から $u_m = \frac{1}{2v_p}\vec{B}\cdot\hat{k}\times\vec{E} = \frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$ を得る。

故に $u = \frac{1}{v_p}\hat{k}\cdot\vec{E}\times\vec{B}$ となる。

以上より，$v_r = \frac{S}{u} = v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$ を得る。

Part B. 一軸性誘電体媒体中の光の伝搬（4.8 点)

B.1　1.5 点

答え：

$n = n_0$，$\hat{B} = \pm\hat{k} \times \hat{y} = \pm(-\cos\theta, 0, \sin\theta)$，$\hat{D} = \pm\hat{y}$

または,

$n = \frac{n_0 n_e}{\sqrt{n_0^2\sin^2\theta + n_e^2\cos^2\theta}}$，$\hat{B} = \pm\hat{y}$，$\hat{D} = \pm\hat{y}\times\hat{k} = \pm(\cos\theta, 0, -\sin\theta)$。

$\theta = 0$ に対して，屈折率に 1 つだけ許される値がある。

解答例：

$\vec{k} \times \vec{E} = \omega\vec{B} = \omega\mu_0\vec{H}$，$\vec{k} \times \vec{H} = -\omega\vec{D}$，より,

$\vec{k}\times(\vec{k}\times\vec{E})=-\omega^2\mu_0\vec{D}$，を得る。各成分を書きだし，$\omega=\dfrac{c}{n}k$ を用いると

$$\begin{aligned}
-\cos^2\theta E_x+\cos\theta\sin\theta E_z &= -\frac{n_0^2}{n^2}E_x\\
-\cos^2\theta E_y-\sin^2\theta E_y &= -\frac{n_0^2}{n^2}E_y\\
-\sin^2\theta E_z+\cos\theta\sin\theta E_x &= -\frac{n_e^2}{n^2}E_z
\end{aligned}$$

少し整理すると

$$\begin{aligned}
\left(1-\frac{n_0^2}{n^2}\right)E_y &= 0\\
\left(\frac{n_0^2}{n^2}-\cos^2\theta\right)E_x+\cos\theta\sin\theta E_z &= 0\\
\cos\theta\sin\theta E_x+\left(\frac{n_e^2}{n^2}-\sin^2\theta\right)E_z &= 0
\end{aligned}$$

行列式を 0 として

$$\left(1-\frac{n_0^2}{n^2}\right)\left[\left(\frac{n_0^2}{n^2}-\cos^2\theta\right)\left(\frac{n_e^2}{n^2}-\sin^2\theta\right)-\cos^2\theta\sin^2\theta\right]=0$$

となる。これより，明らかに一般の θ に対して，n に対する解が 2 つ存在する。

(1) $n=n_0$

この場合は，$E_x=E_z=0$ である。$\vec{E}$ は y 軸に平行である。$\vec{k}\times\vec{E}=\omega\vec{B}$ および $\vec{k}\times(\mu_0\vec{B})=-\omega\vec{D}$ より $\vec{B}$ と $\vec{D}$ の方向は $\hat{B}=\pm\hat{k}\times\hat{y}=\pm(-\cos\theta,0,\sin\theta)$ および $\hat{D}=-\hat{k}\times\hat{B}=\pm(0,1,0)=\pm\hat{y}$ となる。

(2)

$$\left(\frac{n_0^2}{n^2}-\cos^2\theta\right)\left(\frac{n_e^2}{n^2}-\sin^2\theta\right)-\cos^2\theta\sin^2\theta=0$$

これより

$$n=\frac{n_0n_e}{\sqrt{n_0^2\sin^2\theta+n_e^2\cos^2\theta}}$$

が得られる。明らかに，$\theta=0$, $n=n_0$ では屈折率は 1 つのみである。これは光軸の方向である。

この場合，$E_y=0$ で，故に $\vec{E}$ は xz 面内にある。

従って $\vec{k} \times \vec{E} = \omega \vec{B}$ は $\hat{B} = \pm \hat{y}$ を意味する。

また，$\vec{k} \times (\mu_0 \vec{B}) = -\omega \vec{D}$ は $\hat{D} = \pm \hat{y} \times \hat{k}$ を意味する。

B.2　0.8 点

答え：

(1) $n = n_0$ のとき，$\hat{E} = \pm \hat{y}$ で正常光線である。

$\tan \alpha = 0$。

(2)

$$n = \frac{n_0 n_e}{\sqrt{n_0^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}$$

のとき，

$$\hat{E} = \pm \frac{1}{\sqrt{n_0^4 \sin^2 \theta + n_e^4 \cos^2 \theta}} (-n_e^2 \cos \theta, 0, n_0^2 \sin \theta)$$

で異常光線である。

$$\tan \alpha = \frac{(n_0^2 - n_e^2) \tan \theta}{n_e^2 + n_0^2 \tan^2 \theta}$$

解答例：

(1) $n = n_0$ に対しては $\hat{E}$ と $\hat{D}$ は y 軸に対して平行である。これは正常光線であり，$\tan \alpha = 0$ である。

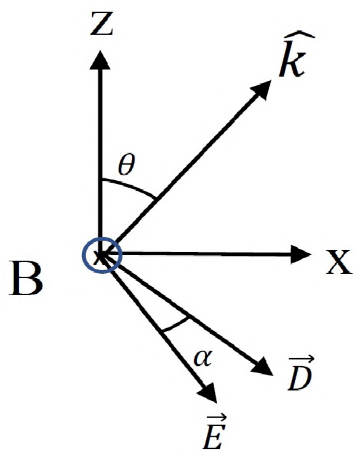
(2)

$$n = \frac{n_0 n_e}{\sqrt{n_0^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}$$

に対して $n \neq n_0$ で，$E_y = 0$ である。

n を E_x と E_z の式に代入すれば，

$$\frac{n_0^2}{n_e^2} \sin \theta E_x + \cos \theta E_z = 0$$



$\vec{D} \perp \hat{k}$

電場は xz 面にあって

$$\hat{E} = \pm \frac{1}{\sqrt{n_0^4 \sin^2\theta + n_e^4 \cos^2\theta}}(-n_e^2\cos\theta, 0, n_0^2\sin\theta)$$

（$\vec{B}$ は $\mp y$ 方向を向いている。）故に $\vec{E}$ は $\vec{k}$ に垂直ではなく，$\vec{D}$ と $\vec{k}$ とともに xz 面内にある。これは異常光線である。

$\vec{k} \times \vec{H} = -\omega\vec{D}$ は $\hat{k}$ に垂直である。故に $\hat{D} = \pm(-\cos\theta, 0, \sin\theta)$。$\vec{B} = \hat{y}$ として与えられた θ に対する $\vec{E}$ と $\vec{D}$ の相対方向は $n_e < n_0$ のときについて図のように示される。

$\vec{E}$ と $\vec{D}$ のそれぞれの x 軸に対する相対角度を θ_1 と θ_2 とする。すると $\tan\theta_2 = -\tan\theta$ および $\tan\theta_1 = -\frac{n_0^2}{n_e^2}\tan\theta$ である。故に

$$\tan\alpha = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1\tan\theta_2} = \frac{(n_0^2 - n_e^2)\tan\theta}{n_e^2 + n_0^2\tan^2\theta}$$

$\vec{E}$ と $\vec{D}$ の相対的な方向が逆転した $\tan\alpha < 0$ をのぞいて $n_e > n_0$ のばあいも同じ結果が保たれる。

B.3　0.6 点

答え：

$n = n_0$ のとき，$\hat{E} = \pm\hat{y}$ で正常光線である。

$n = \dfrac{n_0 n_e}{\sqrt{n_0^2 \sin^2\theta + n_e^2 \cos^2\theta}}$ のとき，

$$\hat{E} = \pm \frac{1}{\sqrt{n_0^4 \sin^2\theta + n_e^4 \cos^2\theta}} \frac{-n_e^2 \cos\theta \hat{k} + (n_0^2 \sin^2\theta - n_e^2 \cos^2\theta)\hat{z}}{\sin\theta}$$

で異常光線である。

解答例：
この問題においては軸対称性があり，z 軸と $\hat{k}$ との平面内で $\vec{k} = k_z\hat{z} + k_\perp\hat{k}_\perp$ および $\vec{E} = E_z\hat{z} + E_\perp\hat{k}_\perp$ と書くことができる。ここで $\hat{k}_\perp$ は $\hat{z}$ に垂直方向を表している。すなわち $k_z = k\cos\theta$，$k_\perp = k\sin\theta$，$E_z = E\cos\theta$，$E_\perp = E\sin\theta$ である。これは

$$\vec{k} \times (\vec{k} \times \vec{E}) = -\omega^2 \mu_0 \vec{D}$$

の成分を書き下せば E_x を $E_\perp$ で置き換えれば同じ関係式を得る。ゆえにすべての解は $\hat{x}$ を $\hat{k}_\perp$ で置き換えれば同じである。$\hat{k}_\perp \sin\theta = \hat{k} - \cos\theta\hat{z}$ なので

$$n = \frac{n_0 n_e}{\sqrt{n_0^2 \sin^2\theta + n_e^2 \cos^2\theta}}$$

のとき答えの結果を得る。

B.4　0.8 点

答え：

(1) $n = n_0$，$\tan\alpha_r = 0$，$v_r = \dfrac{c}{n_0}$，$\hat{S} = (\sin\theta, 0, \cos\theta)$

(2)

$$\begin{aligned} n &= \frac{n_0 n_e}{\sqrt{n_0^2 \sin^2\theta + n_e^2 \cos^2\theta}} \\ \tan\alpha_r &= \frac{(n_0^2 - n_e^2)\tan\theta}{n_e^2 + n_0^2 \tan^2\theta} \\ v_r &= \frac{c}{n_0 n_e}\sqrt{\frac{n_0^4 \sin^2\theta + n_e^4 \cos^2\theta}{n_0^2 \sin^2\theta + n_e^2 \cos^2\theta}} \end{aligned}$$

$$\hat{S} = \frac{1}{\sqrt{n_0^4 \sin^2\theta + n_e^4 \cos^2\theta}} (n_0^2 \sin\theta, 0, n_e^2 \cos\theta)$$

(3) $n_s = \sqrt{(\hat{S}\cdot\hat{x})^2 n_e^2 + (\hat{S}\cdot\hat{z})^2 n_0^2}$

解答例：

エネルギー流の方向はポインティングベクトル $\vec{S} = \vec{E} \times \vec{H}$ である。電磁波のエネルギー密度を u，速さを v_r とする。すなわち $v_r = \frac{S}{u}$ である。ここに $u = u_e + u_m$，$u_e = \frac{1}{2}\vec{E}\cdot\vec{D}$，$u_m = \frac{1}{2}\vec{B}\cdot\vec{H}$ である。以下，2 つの場合がある。

(1) $n = n_0$ の場合，$\vec{E} = (0, E, 0)$，$\vec{D} = \epsilon\vec{E}$，$\vec{k}\times\vec{E} = \omega\mu_0\vec{H}$，$\vec{k}\times\vec{H} = -\omega\vec{D}$。$\hat{k}$，$\vec{E}$，$\vec{H}$ は互いに垂直である。故に $\vec{S}$ は $\hat{k}$ に平行で $\hat{S} = (\sin\theta, 0, \cos\theta)$，$\tan\alpha_r = 0$ である。

$\vec{k}\times\vec{H} = -\omega\vec{D}$ より，$\vec{D} = -\frac{1}{v_p}\hat{k}\times\vec{H}$ を得る。故に $u_e = -\frac{1}{2v_p}\vec{E}\cdot\hat{k}\times\vec{H} = \frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$。同様に $u_m = \frac{1}{2v_p}\vec{H}\cdot\hat{k}\times\vec{E} = \frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$。故に $\frac{1}{v_p}\hat{k}\cdot\vec{E}\times\vec{H}$。故に $\hat{S} = \hat{k}$，また $u = \frac{S}{v_p}$ である。

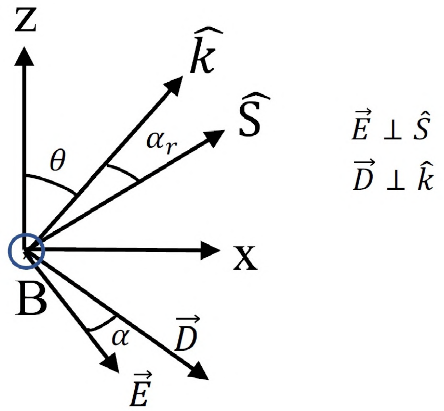
よって $v_r = \frac{S}{u} = v_p = \frac{\omega}{k} = \frac{c}{n_0}$。

(2) $n = \frac{n_0 n_e}{\sqrt{n_0^2 \sin^2\theta + n_e^2 \cos^2\theta}}$ の場合，$\vec{B} = (0, B, 0)$（y 方向が負の場合も同様）ととることができ，$\vec{D}$, $\vec{E}$, $\hat{k}$ は xz 面にあり，$\vec{D}$ は $\hat{k}$ に垂直である。故に $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ と $\hat{k}$ の間の角度は $\vec{D}$ と $\vec{E}$ の間の角と等しい：$\alpha = \alpha_r$。これは $n_e < n_0$ の場合に図のように示される。（$n_e > n_0$ に対しては α, α_r は両者負で $\vec{E}$, $\vec{D}$ の相対方向を逆転させ $\hat{S}$, $\hat{k}$ を入れ替える。）

以上より，$\tan\alpha_r = \tan\alpha = \frac{(n_0^2 - n_e^2)\tan\theta}{n_e^2 + n_0^2 \tan^2\theta}$。

また，$u = \frac{1}{v_p}\hat{k}\cdot\vec{E}\times\vec{H} = \frac{1}{v_p}|\vec{E}\times\vec{H}|\cos\alpha$ 故に $v_r = \frac{S}{u} = \frac{v_p}{\cos\alpha}$ を得る。位相速度 v_p と光速度との関係は $v_p = v_r \cos\alpha$ である。

$\tan\alpha$ より $\cos\alpha = \frac{n_0^2 \sin^2\theta + n_e^2 \cos^2\theta}{\sqrt{n_0^4 \sin^2\theta + n_e^4 \cos^2\theta}}$ である。



故に

$$v_r = \frac{c}{n\cos\alpha} = \frac{c}{n_0 n_e}\sqrt{\frac{n_0^4\sin^2\theta + n_e^4\cos^2\theta}{n_0^2\sin^2\theta + n_e^2\cos^2\theta}}$$

また，$\hat{S} = (\sin(\theta+\alpha), \cos(\theta+\alpha))$ より，

$$\begin{aligned} \sin\alpha &= \frac{(n_0^2 - n_e^2)\sin\theta\cos\theta}{\sqrt{n_0^4\sin^2\theta + n_e^4\cos^2\theta}} \\ \cos\alpha &= \frac{n_e^2\cos^2\theta + n_0^2\sin^2\theta}{\sqrt{n_e^4\cos^2\theta + n_0^4\sin^2\theta}} \end{aligned}$$

また，

$$\hat{S} = \frac{1}{\sqrt{n_0^4\sin^2\theta + n_e^4\cos^2\theta}}(n_0^2\sin\theta, 0, n_e^2\cos\theta)。$$

(3)

$$n_s^2 = \left(\frac{c}{v_r}\right)^2 = n_0^2 n_e^2 \frac{n_e^2\cos^2\theta + n_0^2\sin^2\theta}{n_e^4\cos^2\theta + n_0^4\sin^2\theta} = \frac{n_0^4\sin^2\theta n_e^2 + n_e^4\cos^2\theta n_0^2}{n_e^4\cos^2\theta + n_0^4\sin^2\theta}$$

より

$$n_s = (\hat{S}\cdot\hat{x})^2 n_e^2 + (\hat{S}\cdot\hat{z})^2 n_0^2$$

B.5　1.1 点

答え：

$$\begin{aligned} \bar{A} &= P_1(n^2\sin^2\theta_1 - P_1) \\ \bar{B} &= -2P_3(n^2\sin^2\theta_1 - P_1) \\ \bar{C} &= P_2 n^2\sin^2\theta_1 - P_3^2 \end{aligned}$$

$$\phi = 0,\ \tan\theta_2 = \frac{n n_e \sin\theta_1}{n_0\sqrt{n_0^2 - n^2\sin^2\theta_1}}$$

$$\phi = \pi/2,\ \tan\theta_2 = \frac{n n_0 \sin\theta_1}{n_e\sqrt{n_e^2 - n^2\sin^2\theta_1}}$$

解答例：

A 点と B 点の z 軸に対する距離を d とし，光線が通過する界面の点を原点 O とする。A 点と B 点の座標はそれぞれ $(h_1, 0, d-z)$ 及び$(h_2, 0, z)$ である。$\overline{\mathrm{AO}} \equiv d_1 = \sqrt{h_1^2 + (d-z)^2}$，$\overline{\mathrm{BO}} \equiv d_2 = \sqrt{h_2^2 + z^2}$ である。A から B への伝播時間は光線の速さ v_r によって $(d_1 n_{s1} + d_2 n_{s2})/c$ で決まる。ここに n_{si} は媒質 i の光学定数である。これはフェルマーの原理に従えば，$\varDelta \equiv d_1 n_{s1} + d_2 n_{s2}$ で定義される光路長を極小化する必要がある。これより，$n_{s2}^2 = (\hat{\mathrm{b}}\cdot\hat{x}_2)^2 n_e^2 + (\hat{\mathrm{b}}\cdot\hat{z}_2)^2 n_0^2$ を得る。但し $\hat{\mathrm{b}}$ は $\overrightarrow{\mathrm{OB}}$ の単位ベクトルである。

等方媒質の光学定数は単純な屈折率であり，$n_{s1} = n$ である。次の

$$\begin{aligned}\hat{\mathrm{b}}\cdot\hat{x}_2 &= \cos(\phi - \theta_2) = \frac{h_2}{d_2}\cos\phi + \frac{z}{d_2}\sin\phi \\ \hat{\mathrm{b}}\cdot\hat{z}_2 &= \cos(\frac{\pi}{2} + \phi - \theta_2) = \sin(\theta_2 - \phi) = \frac{z}{d_2}\cos\phi - \frac{h_2}{d_2}\sin\phi\end{aligned}$$

より次を得る。

$$\varDelta = n\sqrt{h_1^2 + (d-z)^2} + \sqrt{(h_2\cos\phi + z\sin\phi)^2 n_e^2 + (-h_2\sin\phi + z\cos\phi)^2 n_0^2}$$

極小値は $\dfrac{\mathrm{d}\varDelta}{\mathrm{d}z} = 0$ により次式が得られる。

$$n\frac{z-d}{\sqrt{h_1^2 + (d-z)^2}} + \frac{h_2\sin\phi\cos\phi(n_e^2 - n_0^2) + z(n_e^2\sin^2\phi + n_0^2\cos^2\phi)}{\sqrt{(h_2\cos\phi + z\sin\phi)^2 n_e^2 + (-h_2\sin\phi + z\cos\phi)^2 n_0^2}} = 0$$

$\dfrac{z-d}{\sqrt{h_1^2 + (d-z)^2}} = \sin\theta_1$ であることなどから，

$$n^2\sin^2\theta_1 = \frac{(P_3 - P_1\tan\theta_2)^2}{P_1\tan^2\theta_2 - 2P_3\tan\theta_2 + P_2}$$

を得る。ただし，

$$\begin{aligned}P_1 &= n_0^2\cos^2\phi + n_e^2\sin^2\phi \\ P_2 &= n_0^2\sin^2\phi + n_e^2\cos^2\phi \\ P_3 &= (n_0^2 - n_e^2)\sin\phi\cos\phi\end{aligned}$$

以上から

$$P_1(n^2\sin^2\theta_1 - P_1)\tan^2\theta_2 - 2P_3(n^2\sin^2\theta_1 - P_1)\tan\theta_1 + P_2 n^2\sin^2\theta_1 - P_3^2 = 0$$

となる。これより

$$\begin{aligned}\bar{A} &= P_1(n^2\sin^2\theta_1 - P_1)\\ \bar{B} &= -2P_3(n^2\sin^2\theta_1 - P_1)\\ \bar{C} &= P_2 n^2\sin^2\theta_1 - P_3^2\end{aligned}$$

$\phi = 0$ に対しては，$P_3 = 0$，$P_1 = n_0^2$，$P_2 = n_e^2$。
$n_0^2(n^2\sin^2\theta_1 - n_0^2)\tan^2\theta_2 + n_e^2 n^2\sin^2\theta_1 = 0$ より，

$$\tan\theta_2 = \frac{n n_e \sin\theta_1}{n_0\sqrt{n_0^2 - n^2\sin^2\theta_1}}$$

$\phi = \pi/2$ に対しては，$P_3 = 0$，$P_1 = n_e^2$，$P_2 = n_0^2$。
$n_e^2(n^2\sin^2\theta_1 - n_e^2)\tan^2\theta_2 + n_0^2 n^2\sin^2\theta_1 = 0$ より

$$\tan\theta_2 = \frac{n n_0 \sin\theta_1}{n_e\sqrt{n_e^2 - n^2\sin^2\theta_1}}$$

Part C. 光のもつれ（エンタングルメント）(4.2 点)

C.1　0.8 点

答え：

(1) $\omega = \omega_1 \pm \omega_2$，$\vec{k} = \vec{k}_1 \pm \vec{k}_2$

(2) $\hbar\omega = \hbar\omega_1 \pm \hbar\omega_2$，光子のエネルギー保存
$\hbar\vec{k} = \hbar\vec{k}_1 \pm \hbar\vec{k}_2$，光子の運動量保存

(3) 光子の分裂：エネルギー保存に対して $\omega = \omega_1 + \omega_2$，
運動量保存に対して $\vec{k} = \vec{k}_1 + \vec{k}_2$

解答例：
角振動数 ω，波数ベクトル $\vec{k}$ の光波に対してはその電場は $\vec{A}\cos(\omega t - \vec{k}\cdot\vec{r})$ のように与えられる。これはまた，$\dfrac{\vec{A}}{2}(e^{i(\omega t-\vec{k}\cdot\vec{r})} + e^{-i(\omega t-\vec{k}\cdot\vec{r})})$ と書き直すことができる。この式を

$$P_i^{NL} = \sum_j \sum_k \chi_{ijk}^{(2)} E_j E_k$$

に代入し，指数の等式を整理すると

$$\begin{aligned} \omega &= \omega_1 + \omega_2, \qquad \vec{k} = \vec{k}_1 + \vec{k}_2 \quad \text{または} \\ \omega &= \omega_1 - \omega_2, \qquad \vec{k} = \vec{k}_1 - \vec{k}_2 \end{aligned}$$

となる。ただし，振動数は正である。この関係は光子のエネルギー $\hbar\omega$ と運動量 $\hbar\vec{k}$ から明らかである。$\omega = \omega_1 + \omega_2$，$\vec{k} = \vec{k}_1 + \vec{k}_2$ は $(\omega, \vec{k})$ の光子が消滅して 2 つの光子 $(\omega_1, \vec{k}_1)$ と $(\omega_2, \vec{k}_2)$ に分裂した。一方 $\omega = \omega_1 - \omega_2$，$\vec{k} = \vec{k}_1 - \vec{k}_2$ は $(\omega_1, \vec{k}_1)$ の光子が消滅して 2 つの光子 $(\omega, \vec{k})$ と $(\omega_2, \vec{k}_2)$ に分裂した。

C.2　0.8 点

答え：

o→**o** ＋ **o**，　　**e**→**e** ＋ **e**

解答例：
同一直線上で位相の整合条件は

$$\omega = \omega_1 + \omega_2 \qquad \frac{n_i(\omega)\omega}{c} = \frac{n_j(\omega_1)\omega_1}{c} + \frac{n_k(\omega_2)\omega_2}{c}$$

である。ここに i, j, k は **o** か **e** かである。$\omega_1 \geq \omega_2$ であると仮定すると，ω_1 は $\omega_1 = \omega - \omega_2$ として解くことができ，次を得る。

$$n_i(\omega) - n_j(\omega_1) = \frac{\omega_2}{\omega}\{n_k(\omega_2) - n_j(\omega_1)\}$$

これは，もし，$i = j = k$ で $n_i(\omega) - n_j(\omega_1) \geq 0$，かつ $n_k(\omega_2) - n_j(\omega_1) \leq 0$ であれば，$\omega \geq \omega_1 \geq \omega_2$，であることから，上記の方程式は満たされないことは明かである。
他の場合に関しては n_0 と n_e には関係が無いので，位相の整合条件は満たされている。故に，$\omega = \omega_1 \pm \omega_2$，$\vec{k} = \vec{k}_1 \pm \vec{k}_2$ は不可能である。故に **o**→**o** ＋ **o**，**e**→**e** ＋ **e** のみ不可能である。

C.3　1.3 点

答え：

(1)

$$\begin{aligned} M &= \frac{K_0\{1 - N_e(\Omega_e, \theta)\cot\theta\} + K_e}{2K_eK_0} \\ N &= -\frac{N_e}{2M} \\ L &= -(\Omega - \Omega_e)\left(\frac{1}{u_0} - \frac{1}{u_e}\right) + \frac{N_e^2}{4M} \end{aligned}$$

(2) 円錐の軸と z' の間の角度 $\tan^{-1}\left(\frac{N}{K_0}\right)$ は

$$\frac{N}{K_0} = -\frac{2K_eN_e}{K_0\{1 - N_e(\Omega_e, \theta)\cot\theta\} + K_e}$$

(3) 円錐の角度 $\tan^{-1}\left(\frac{\sqrt{L/M}}{K_0}\right)$ は

$$\frac{\sqrt{L/M}}{K_0} = -\frac{\Omega - \Omega_e}{MK_0}\left(\frac{1}{u_o} - \frac{1}{u_e}\right) + \frac{N_e^2}{4M^2K_0}$$

解答例：

位相の整合性を満たすために角周波数 ω_1 と ω_2 を $\omega_1 = \Omega_e + \nu$ と $\omega_2 = \Omega_0 + \nu'$ として，新たな ν と ν' で展開する。

$\Omega_e + \Omega_0 = \Omega_p$ より $\omega_1 + \omega_2 = \omega$ ，$\nu = -\nu'$ である。

同じように波数ベクトルに対しても $\vec{k} = \vec{k}_1 + \vec{k}_2$ の条件は $k_z = k = K_p = k_{1z} + k_{2z}$ および $\vec{k}_{2\perp} = -\vec{k}_{1\perp} \equiv \vec{q}_\perp$ のように書くことが出来る。

通常光線 **o** に対しては $k_{2\perp}^2 + k_{2z}^2 = k_2^2$，$k_2 = \frac{n_0(\omega_2)\omega_2}{c}$ である。これより $k_{2z} = \sqrt{k_2^2 - k_{2\perp}^2} = k_2 - \frac{k_{2\perp}^2}{2k_2}$ となる。k_2 の ω_2 依存性の ν に対する展開をすると

$$k_2 = \frac{n_0(\omega_2)\omega_2}{c} = \frac{n_0(\Omega_0)\Omega_0}{c} + \frac{\mathrm{d}k_2}{\mathrm{d}\omega_2}(\omega_2 - \Omega_0) = K_0 - \frac{\nu}{u_0}$$

を得る。ここで u_0 は通常光線の速さである。故に 2 次までに関して

$$k_{2z} = K_0 - \frac{\nu}{u_0} - \frac{q_\perp^2}{2K_0}$$

を得る。同様に異常光線 e に対しては $k_{1\perp}^2 + k_{1z}^2 = k_1^2$，$k_1 = \dfrac{n_e(\omega_1, \theta_1)\omega_1}{c}$ である。これより $k_{1z} = \sqrt{k_1^2 - k_{1\perp}^2} = k_1 - \dfrac{k_{1\perp}^2}{2k_1}$ となる。k_1 の展開は，k_2 とはその角度依存性に関して異なる。$\vec{k}_1$ の球面角を θ_1 と ϕ_1 とする。これより

$$k_1 = \frac{n_e(\omega_1, \theta_1)\omega_1}{c} = \frac{n_e(\Omega_e, \theta)\Omega_e}{c} + \frac{\mathrm{d}k_1(\Omega_e, \theta)}{\mathrm{d}\Omega_e}(\omega_1 - \Omega_e) + \frac{\Omega_e}{c}\frac{\mathrm{d}n_e(\Omega_e, \theta)}{\mathrm{d}\theta}(\theta_1 - \theta) + \cdots$$

ここに $\dfrac{n_e(\Omega_e, \theta)\Omega_e}{c} = K_e$，また，$\dfrac{\mathrm{d}k_1(\Omega_e, \theta)}{\mathrm{d}\Omega_e}$ は異常光線の速さ u_e に対して $1/u_e$ であり

$$\frac{\mathrm{d}k_1(\Omega_e, \theta)}{\mathrm{d}\Omega_e} = \frac{n_e(\Omega_e, \theta)}{c} + \frac{\Omega_e}{c}\frac{\mathrm{d}n_e(\Omega_e, \theta)}{\mathrm{d}\Omega_e}$$

と得られる。さらに，

$$\frac{\mathrm{d}n_e(\Omega_e, \theta)}{\mathrm{d}\theta} = \frac{n_e n_o(n_e^2 - n_0^2)\sin\theta\cos\theta}{(n_0^2\sin^2\theta + n_e^2\cos^2\theta)^{3/2}} = n_e(\Omega_e, \theta)N_e(\Omega_e, \theta)$$

であることから

$$N_e(\Omega_e, \theta) = \frac{(n_e^2 - n_0^2)\sin\theta\cos\theta}{n_0^2\sin^2\theta + n_e^2\cos^2\theta}$$

を得る。$n_e < n_0$ に対して $N_e(\Omega_e, \theta) < 0$ であることに注意しよう。
$\delta\theta = \theta_1 - \theta$ を求めるために，任意の $\vec{k}_\alpha$ に対して

$$\hat{k}_\alpha \cdot \widehat{\mathrm{OA}} = \cos\theta_\alpha = \cos\theta\cos\psi_\alpha + \sin\theta\sin\psi_\alpha\cos\phi_\alpha$$

を得る（問題の図 2(a) 参照）。
$\sin\psi_1 = |\vec{k}_{\perp,1}|/|\vec{k}_1| = q_\perp/k_1 \ll 1$ であり，　また，　$\cos\psi_1 = \sqrt{1 - \sin^2\psi_1}$ $= 1 - (1/2)\sin^2\psi_1 + \cdots$ の 2 次までで，k_1 を K_e で置き換え

$$\hat{k}_1 \cdot \widehat{\mathrm{OA}} = \cos\theta_1 = \cos\theta\left(1 - \frac{1}{2}\frac{q_\perp^2}{K_e^2} + \cdots\right) + \sin\theta\left(\frac{q_\perp}{K_e} + \cdots\right)\cos\phi_1$$

を得る。

一方，$\cos\theta_1 = \cos\theta + \dfrac{\mathrm{d}\cos\theta}{\mathrm{d}\theta}(\theta_1-\theta)+\cdots = \cos\theta - \sin\theta(\theta_1-\theta)+\cdots$ であり，これと $\hat{k}_1\cdot\widehat{\mathrm{OA}}$ の方程式と比べて

$$\theta_1-\theta = \frac{1}{2}\frac{q_\perp^2}{K_e^2}\cot\theta - \frac{q_\perp}{K_e}\cos\phi_1+\cdots = \frac{1}{2}\frac{q_\perp^2}{K_e^2}\cot\theta + \frac{q_{x'}}{K_e}+\cdots$$

以上をすべて合わせて

$$k_{1z} = K_e + \frac{1}{u_e}(\Omega-\Omega_e) + N_e(\Omega_e,\theta)q_{x'} + \frac{1}{2}\frac{q_\perp^2}{K_e}[N_e(\Omega_e,\theta)\cot\theta - 1]+\cdots$$

となることが分かる。この式を k_{1z} と $K_p = k_{1z}+k_{2z}$ と合わせると

$$(\Omega-\Omega_e)\left(\frac{1}{u_e}-\frac{1}{u_0}\right) + N_e(\Omega_e,\theta)q_{x'} + q_\perp^2\left\{\frac{K_0[N_e(\Omega_e,\theta)\cot\theta-1]-K_e}{2K_eK_0}\right\} = 0$$

を得る。$n_e < n_0$ より $N_e(\Omega_e,\theta) < 0$ である。これより

$$M\left(q_{x'} - \frac{N_e}{2M}\right)^2 + Mq_{y'}^2 = -(\Omega-\Omega_e)\left(\frac{1}{u_0}-\frac{1}{u_e}\right) + \frac{N_e^2}{4M}$$

と書きかえることが出来る。ここに

$$M = -\frac{K_0[N_e(\Omega_e,\theta)\cot\theta-1]-K_e}{2K_eK_0} > 0$$

である。

故に $N = -N_e/2M > 0$ $(N_e < 0)$，および，

$$L = -(\Omega-\Omega_e)\left(\frac{1}{u_0}-\frac{1}{u_e}\right) + \frac{N_e^2}{4M}$$

である。明らかに $\vec{k}_2$ で形成される円錐の軸は $\vec{q}_\perp$ で特徴づけられる。円錐の軸と z' の間の角度は $\tan^{-1}(N/k_{1z})$ であり，これは

$$N/k_{1z} \approx \frac{N}{K_0} = -\frac{2K_eN_e}{K_0\{1-N_e(\Omega_e,\theta)\cot\theta\}+K_e}$$

である。円錐の角度は

$$\sin^{-1}\frac{\sqrt{L/M}}{k_0} \approx \frac{\sqrt{L/M}}{K_0} = -\frac{\Omega-\Omega_e}{MK_0}\left(\frac{1}{u_o}-\frac{1}{u_e}\right) + \frac{N_e^2}{4M^2K_0}$$

である。

C.4　0.8 点

答え：

$$\begin{aligned} P(\alpha,\beta) &= \frac{1}{2}\sin^2(\alpha+\beta) \\ P(\alpha,\beta_\perp) &= \frac{1}{2}\cos^2(\alpha+\beta) \\ P(\alpha_\perp,\beta) &= \frac{1}{2}\cos^2(\alpha+\beta) \\ P(\alpha_\perp,\beta_\perp) &= \frac{1}{2}\sin^2(\alpha+\beta) \end{aligned}$$

解答例：

a 光子に対して偏光の方向の電場と偏光に垂直の電場をそれぞれ $|\alpha_x\rangle$ と $|\alpha_y\rangle$ で表すことにしよう。ここに α_x および α_y は適当な単位で計った電場の振幅を表すものとする。$\hat{x}'$ と $\hat{y}'$ 方向の電場（の状態）は

$$\begin{aligned} |\hat{x}'_a\rangle &= \cos\alpha|\alpha_x\rangle - \sin\alpha|\alpha_y\rangle \\ |\hat{y}'_a\rangle &= \sin\alpha|\alpha_x\rangle + \cos\alpha|\alpha_y\rangle \end{aligned}$$

同様に b 光子に対して

$$\begin{aligned} |\hat{x}'_b\rangle &= \cos\beta|\beta_x\rangle - \sin\beta|\beta_y\rangle \\ |\hat{y}'_b\rangle &= \sin\beta|\beta_x\rangle + \cos\beta|\beta_y\rangle \end{aligned}$$

である。これらから

$$\begin{aligned} |\hat{x}'_a\rangle|\hat{y}'_b\rangle &= (\cos\alpha|\alpha_x\rangle - \sin\alpha|\alpha_y\rangle)(\sin\beta|\beta_x\rangle + \cos\beta|\beta_y\rangle) \\ |\hat{y}'_a\rangle|\hat{x}'_b\rangle &= (\sin\alpha|\alpha_x\rangle + \cos\alpha|\alpha_y\rangle)(\cos\beta|\beta_x\rangle - \sin\beta|\beta_y\rangle) \end{aligned}$$

もつれた光子対に対しては

$$\begin{aligned} & \frac{1}{\sqrt{2}}(|\hat{x}'_a\rangle)|\hat{y}'_b\rangle + |\hat{y}'_a\rangle|\hat{x}'_b\rangle) \\ = & \frac{1}{\sqrt{2}}\{(\cos\alpha\sin\beta + \sin\alpha\cos\beta)(|\alpha_x\rangle|\beta_x\rangle - |\alpha_y\rangle)|\beta_y\rangle) \\ & +(\cos\alpha\cos\beta - \sin\alpha\sin\beta)(|\alpha_x\rangle|\beta_y\rangle - |\alpha_y\rangle)|\beta_x\rangle)\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}\{\sin(\alpha+\beta)(|\alpha_x\rangle|\beta_x\rangle - |\alpha_y\rangle|\beta_y\rangle) \\
&\quad + \cos(\alpha+\beta)(|\alpha_x\rangle|\beta_y\rangle - |\alpha_y\rangle|\beta_x\rangle)\}
\end{aligned}
$$

以上より

$$
\begin{aligned}
P(\alpha,\beta) &= \frac{1}{2}\sin^2(\alpha+\beta) \\
P(\alpha_\perp,\beta_\perp) &= \frac{1}{2}\sin^2(\alpha+\beta) \\
P(\alpha,\beta_\perp) &= \frac{1}{2}\cos^2(\alpha+\beta) \\
P(\alpha_\perp,\beta) &= \frac{1}{2}\cos^2(\alpha+\beta)
\end{aligned}
$$

C.5　0.5 点

答え：

$$S = |\cos 2(\alpha-\beta) - \cos 2(\alpha-\beta')| + |\cos 2(\alpha'-\beta) + \cos 2(\alpha'-\beta')|$$

$S = 2\sqrt{2}$。

$S > 2$ は古典理論とは相容れない。

解答例：

まず気づくのは，

$$E(\alpha,\beta) = \frac{P(\alpha,\beta) + P(\alpha_\perp,\beta_\perp) - P(\alpha,\beta_\perp) - P(\alpha_\perp,\beta)}{P(\alpha,\beta) + P(\alpha_\perp,\beta_\perp) + P(\alpha,\beta_\perp) + P(\alpha_\perp,\beta)}$$

である。P の表式を用いれば

$$
\begin{aligned}
E(\alpha,\beta) &= \sin^2(\alpha+\beta) - \cos^2(\alpha+\beta) \\
&= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)^2 - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)^2 \\
&= -(\cos^2\alpha - \sin^2\alpha)(\cos^2\beta - \sin^2\beta) + 4\sin\alpha\sin\beta\cos\alpha\cos\beta \\
&= \sin(2\alpha)\sin(2\beta) - \cos(2\alpha)\cos(2\beta) \\
&= -\cos 2(\alpha-\beta)
\end{aligned}
$$

故に

$$S = |\cos 2(\alpha-\beta) - \cos 2(\alpha-\beta')| + |\cos 2(\alpha'-\beta) + \cos 2(\alpha'-\beta')|$$

$$\alpha = \frac{\pi}{4}, \quad \alpha' = 0, \quad \beta = -\frac{\pi}{8}, \quad \beta' = \frac{\pi}{8}$$

に対して S は

$$S = \left| -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right| + \left| \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right| = 2\sqrt{2} > 2$$

となる。以上から古典理論とは相容れない。

Theory 3 Magnetic Levitation: Solution

Part A. Sudden appearance of a magnetic monopole: initial response and subsequent time evolution of the response in the thin film

Initial response

A.1 In the $z \geq 0$ region, excluding the point occupied by the monopole, the magnetic field $\vec{B} = \vec{B}' + \vec{B}_{\text{mp}}$ at $t = t_0 = 0$ is given by

$$\vec{B}_{\text{mp}} = \frac{\mu_0 q_{\text{m}}}{4\pi} \frac{(z-h)\hat{z}+\vec{\rho}}{[(z-h)^2+\rho^2]^{3/2}}, \quad \text{(A-1)}$$

$$\vec{B}' = \frac{\mu_0 q_{\text{m}}}{4\pi} \frac{(z+h)\hat{z}+\vec{\rho}}{[(z+h)^2+\rho^2]^{3/2}}, \quad \text{(A-2)}$$

$$\boxed{\vec{B} = \frac{\mu_0 q_{\text{m}}}{4\pi} \left[\frac{(z-h)\hat{z}+\vec{\rho}}{[(z-h)^2+\rho^2]^{3/2}} + \frac{(z+h)\hat{z}+\vec{\rho}}{[(z+h)^2+\rho^2]^{3/2}} \right].} \quad \text{(A-3)}$$

A.2 In the $z \leq -d$ region, the magnetic field $\vec{B} = \vec{B}' + \vec{B}_{\text{mp}}$ at $t = t_0 = 0$ is given by

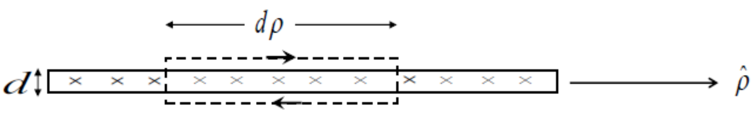
$$\boxed{\vec{B} = 0.} \quad \text{(A-4)}$$

A.3 From Eq. (A-3), $B_z' = 0$ at $z = 0$ for all ρ.
Therefore, the total magnetic flux $\Phi_{\text{B}} = 0$ at $z = 0$. (A-5)
From Eq. (A-4), $B_z' = 0$ at $z = -d$.
Therefore, the total magnetic flux $\Phi_{\text{B}} = 0$ at $z = -d$. (A-6)

A.4 Applying Ampere's law along the path shown in the figure below, and using the approximation $d \ll h$, we have

$$B_\rho(\rho,\ z = 0) d\rho = \mu_0\, j(\rho)\, d\rho \cdot d, \quad \text{(A-7)}$$

where the contributions from the $B_z d$ terms are smaller by a factor d/h and neglected.



The induced current density is given by

$$\boxed{\vec{j}(\vec{\rho}) = \frac{1}{\mu_0 d} \hat{z} \times \vec{B}(\vec{\rho}, z = 0) = \frac{q_{\text{m}}}{2\pi d} \frac{\hat{z}\times\vec{\rho}}{(h^2+\rho^2)^{3/2}}.} \quad \text{(A-8)}$$

Subsequent response

A.5 Consider the form of an integral of Eq.(2), in the Question sheet, over the film thickness, we get, for $z \approx 0$ inside the film (that is $z < 0$ and $|z| \ll d$), that

$$\left.\frac{\partial B_z'}{\partial z}\right|_z - \left.\frac{\partial B_z'}{\partial z}\right|_{-d-z} = \mu_0 \sigma (d + 2z) \frac{\partial B_z'}{\partial t} \approx \mu_0 \sigma d \frac{\partial B_z'}{\partial t}. \quad \text{(A-9)}$$

Since B'_z is an even function of $z' = z + d/2$, therefore we have $\left.\frac{\partial B'_z}{\partial z}\right|_z = -\left.\frac{\partial B'_z}{\partial z}\right|_{-d-z}$ so that the left-hand side of Eq.(A-9) becomes $2\frac{\partial}{\partial z}B'_z(\rho, z; t)$. The right-hand side is approximated by the z-independent term of B'_z inside the film thickness. On the other hand, the z-dependent term of B'_z is even in z' and is of order $\sim z'^2 d/h$ so that it can be neglected based on the $h \gg d$ condition. As such the right-hand side is represented by $B'_z(\rho, z; t)$. Putting these results together, we get

$$2\frac{\partial}{\partial z}B'_z(\rho, z; t) = \mu_0 \sigma d \frac{\partial}{\partial t}B'_z(\rho, z; t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}B'_z(\rho, z; t) = v_0 \frac{\partial}{\partial z}B'_z(\rho, z; t).} \qquad \text{(A-10)}$$

Here $z \approx 0$, and $v_0 = 2/(\mu_0 \sigma d)$.

A.6 The equation in **A.5**, namely, Eq.(A-10) supports a solution of the form

$$\boxed{B'_z(\rho, z; t) = f(\rho, z + v_0 t),} \qquad \text{(A-11)}$$

and at $z \approx 0$.

A.7 At $t = 0$, $B'_z(\rho, z \geq 0) = \frac{\mu_0 q_m}{4\pi}\frac{(z+h)}{[(z+h)^2+\rho^2]^{3/2}}$, which is of the form

$$\boxed{B'_z(\rho, z \geq 0) = F(\rho, z + h).} \qquad \text{(A-12)}$$

For $t > 0$, we have according to Eq.(A-11), the replacement

$\boxed{z \to z + v_0 t}$, to the $B'_z(\rho, z; t = 0)$. (A-13)

In other words, $B'_z(\rho, z \approx 0; t) = F(\rho, z + v_0 t + h)$.

This corresponds to a physical picture of a moving image monopole, with its position

$$\boxed{z_{\rm mp} = -h - v_0 t.} \qquad \text{(A-14)}$$

Finally, $\boxed{v_0 = 2/(\mu_0 \sigma d).}$ (A-15)

Part B. Magnetic force acting on a point-like magnetic dipole moving at a constant h with a constant velocity

A moving monopole

B.1 The present locations of all the image magnetic monopoles of type $q_{\rm m}$ are at

$$\boxed{(x, z) = [-nv\tau, -h - nv_0\tau], \text{ for } n \geq 0.} \qquad \text{(B-1)}$$

The locations of all the image magnetic monopoles $-q_{\mathrm{m}}$ are at

$$\boxed{(x,z)=[-(n+1)v\tau,-h-nv_0\tau\,],\ \text{for}\ \ n\geq 0.} \tag{B-2}$$

B.2 The magnetic potential $\Phi_+(x,z)$ due to all the image magnetic monopoles at $t=0$ is given by, in summation form

$$\Phi_+(x,z)=\frac{\mu_0 q_m}{4\pi}\sum_{n=0}^{\infty}\frac{1}{\sqrt{(x+nv\tau)^2+(z+h+nv_0\tau)^2}}-\frac{\mu_0 q_m}{4\pi}\sum_{n=0}^{\infty}\frac{1}{\sqrt{(x+(n+1)v\tau)^2+(z+h+nv_0\tau)^2}},$$

$$\Rightarrow \boxed{\Phi_+(x,z)=\frac{\mu_0 q_m}{4\pi}\sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{(x+nv\tau)^2+(z+h+nv_0\tau)^2}}-\frac{1}{\sqrt{(x+(n+1)v\tau)^2+(z+h+nv_0\tau)^2}}\right].} \tag{B-3}$$

In integral form

$$\Phi_+(x,z)=\frac{\mu_0 q_m}{4\pi\tau}\int_0^{\infty}dt'\left[\frac{1}{\sqrt{(x+vt')^2+(z+h+v_0t')^2}}-\frac{1}{\sqrt{(x+vt'+v\tau)^2+(z+h+v_0t')^2}}\right], \tag{B-4}$$

$$=\frac{\mu_0 q_m}{4\pi\tau}\int_0^{\infty}dt'\,\frac{(x+vt')v\tau}{[(x+vt')^2+(z+h+v_0\tau)^2]^{3/2}}, \tag{B-5}$$

$$\Rightarrow \boxed{\Phi_+(x,z)=\frac{\mu_0 q_m v}{4\pi}\,\frac{1}{(z+h)v-v_0x}\left[\frac{z+h}{\sqrt{x^2+(z+h)^2}}-\frac{v_0}{\sqrt{v^2+v_0^2}}\right].} \tag{B-6}$$

A moving dipole

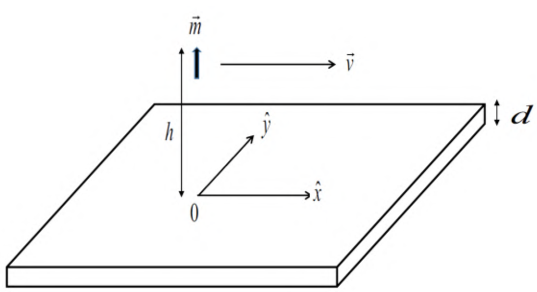
B.3

The total magnetic potential

$$\Phi_{\mathrm{T}}(x,z)=\Phi_+(x,z)+\Phi_-(x,z), \tag{B-7}$$

where $\Phi_-(x,z)=-\Phi_+(x,z-\delta_{\mathrm{m}})$.

$$\Phi_{\mathrm{T}}(x,z)=\Phi_+(x,z)-\Phi_+(x,z-\delta_{\mathrm{m}})$$
$$=\delta_{\mathrm{m}}\times\partial\Phi_+(x,z)/\partial z. \tag{B-8}$$



$$\Phi_{\mathrm{T}}(x,z) = -\frac{\mu_0 m v}{4\pi}\left[\frac{v}{[(z+h)v-v_0x]^2}\left(\frac{z+h}{\sqrt{x^2+(z+h)^2}} - \frac{v_0}{\sqrt{v^2+v_0^2}}\right) - \frac{x^2}{[(z+h)v-v_0x][x^2+(z+h)^2]^{3/2}}\right]. \quad \text{(B-9)}$$

Force acting on the point-like magnetic dipole:

$$F_z = -q_m \frac{d}{dz}\Phi_{\mathrm{T}}(0,z)\Big|_{z=h} + q_m \frac{d}{dz}\Phi_{\mathrm{T}}(0,z)\Big|_{z=h-\delta_{\mathrm{m}}}. \quad \text{(B-10)}$$

$$F_z = -\frac{\mu_0 m q_{\mathrm{m}}}{2\pi}\left(1 - \frac{v_0}{\sqrt{v^2+v_0^2}}\right)\left[\frac{1}{(2h)^3} - \frac{1}{(2h-\delta_{\mathrm{m}})^3}\right]. \quad \text{(B-11)}$$

$$\Rightarrow \quad \boxed{F_z = \frac{3\mu_0 m^2}{32\pi h^4}\left[1 - \frac{v_0}{\sqrt{v^2+v_0^2}}\right].} \quad \text{(B-12)}$$

$$F_x = -q_m \frac{d}{dx}\Phi_{\mathrm{T}}(x,h)\Big|_{x=0} + q_m \frac{d}{dx}\Phi_{\mathrm{T}}(x,h-\delta_{\mathrm{m}})\Big|_{x=0}, \quad \text{(B-13)}$$

$$\Rightarrow \quad \boxed{F_x = -\frac{3\mu_0 m^2}{32\pi h^4}\frac{v_0}{v}\left[1 - \frac{v_0}{\sqrt{v^2+v_0^2}}\right].} \quad \text{(B-14)}$$

Relation between v_0 and v and their relation

B.4 $$\boxed{v_0 = \frac{2}{\mu_0 \sigma d} = \frac{2}{4\pi\times10^{-7}\times5.9\times10^{7}\times0.5\times10^{-2}} = 5.4\ \mathrm{m/s}\ .} \quad \text{(B-15)}$$

B.5 In the small v regime, meaning that v is smaller than a certain typical velocity of the system (or a critical velocity v_{c} to be considered in the next task **B.**6) we have the characteristics basically akin to that of $v \approx 0$. For $v = 0$, the frequency ω is associated with v_0/h. Making use of the parameters given in **B.4**, the skin depth (Eq.(3) in the question sheet) δ is given by

$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} = \sqrt{\frac{2h}{v_0\mu_0\sigma}} = 1.58$ c.m., which is more than three times greater than d.

Thus we have, in the small v regime,

$$\boxed{v_0(v) = v_0.} \quad \text{(B-16)}$$

In the large v regime, we have the skin depth $\delta < d$ so that the effect thin film thickness

$$d_{\text{eff}} = \delta, \tag{B-17}$$

within which the field is more or less uniform (i.e. z independent).
In this case, $\omega = v/h$, (B-18)
so the

$$v_0(v) = \frac{2}{\mu_0\sigma\delta} = \frac{2}{\mu_0\sigma}\sqrt{\frac{\omega\mu_0\sigma}{2}} = \sqrt{\frac{2}{\mu_0\sigma}\frac{v}{h}} = \sqrt{\frac{d}{h}v\, v_0}, \quad \text{or}$$

$$\boxed{v_0(v) = v_0\sqrt{\frac{d}{h}}\sqrt{\frac{v}{v_0}}} \tag{B-19}$$

B.6 The critical velocity v_{c} is determined from the condition $\delta = d$:

$$d = \sqrt{\frac{2}{\mu_0\sigma v_{\text{c}}/h}} \;\Rightarrow\; \boxed{v_{\text{c}} = \frac{2h}{d^2\mu_0\sigma} = v_0\frac{h}{d}} \;. \tag{B-20}$$

Part C Motion of the magnetic dipole when the conducting thin film is superconducting

When the electrical conductivity $\sigma \to \infty$, the receding velocity $v_0 \to 0$ so that there will not be a whole series of image magnetic monopoles. Instead, the image is simply one image magnetic dipole mirroring the instantaneous position of the magnetic dipole. In this case, the image magnetic dipole is $\vec{m} = m\hat{x}$ located at the location $(x, y, z) = (0,0,-h)$. It is then clear, from the symmetry of the image configuration, that the force on the magnetic dipole from the image aligns only along $\hat{z}$. For our convenience, we take the magnetic monopole $-q_{\text{m}}$ to locate at $x = 0$, and for the magnetic monopole q_{m} the location $x = \delta_{\text{m}}$.

C.1

The total magnetic potential $\Phi_{\text{T}}(x, z)$ from the image magnetic dipole is

$$\Phi_{\text{T}}(x, z) = -\frac{\mu_0 q_{\text{m}}}{4\pi}\frac{1}{\sqrt{x^2+(z+h)^2}} + \frac{\mu_0 q_{\text{m}}}{4\pi}\frac{1}{\sqrt{(x-\delta_{\text{m}})^2+(z+h)^2}}. \tag{C-1}$$

Approach 1:

The total vertical force F_z' acting on the magnetic dipole from the image magnetic dipole is given by

$$F_z' = (-q_{\text{m}})\left[-\frac{\partial}{\partial z}\Phi_{\text{T}}\right]\Bigg|_{\substack{x=0,\\ z=h}} + q_{\text{m}}\left[-\frac{\partial}{\partial z}\Phi_{\text{T}}\right]\Bigg|_{\substack{x=\delta,\\ z=h}} \tag{C-2}$$

$$F_z' = \frac{\mu_0 q_{\mathrm{m}}^2}{4\pi} \frac{z+h}{[x^2+(z+h)^2]^{3/2}}\bigg|_{\substack{x=0,\\ z=h}} - \frac{\mu_0 q_{\mathrm{m}}^2}{4\pi} \frac{z+h}{[(x-\delta_{\mathrm{m}})^2+(z+h)^2]^{3/2}}\bigg|_{\substack{x=0,\\ z=h}}$$

$$- \frac{\mu_0 q_{\mathrm{m}}^2}{4\pi} \frac{z+h}{[x^2+(z+h)^2]^{3/2}}\bigg|_{\substack{x=\delta_{\mathrm{m}},\\ z=h}} + \frac{\mu_0 q_{\mathrm{m}}^2}{4\pi} \frac{z+h}{[(x-\delta_{\mathrm{m}})^2+(z+h)^2]^{3/2}}\bigg|_{\substack{x=\delta_{\mathrm{m}},\\ z=h}} \,,$$

$$F_z' = 2\frac{\mu_0 q_{\mathrm{m}}^2}{4\pi}\left(\frac{1}{2h}\right)^2\left[1 - \frac{1}{\left(1+\left(\frac{\delta}{2h}\right)^2\right)^{3/2}}\right] . \qquad \text{(C-3)}$$

$$F_z' = \frac{3\mu_0 m^2}{64\pi h^4}. \qquad \text{(C-4)}$$

Equilibrium condition:

$$F_z' - M_0 g = 0, \qquad \text{(C-5)}$$

$$\Rightarrow \quad \frac{3\mu_0 m^2}{64\pi h_0^4} = M_0 g,$$

$$\Rightarrow \quad \boxed{h_0 = \left[\frac{3\mu_0 m^2}{64\pi M_0 g}\right]^{\frac{1}{4}}.} \qquad \text{(C-6)}$$

Approach 2:

We can use the direct force calculation.

$$F_z' = 2\frac{\mu_0 q_{\mathrm{m}}^2}{4\pi}\left[\left(\frac{1}{2h}\right)^2 - \frac{2h}{\left(\delta_m^2+(2h)^2\right)^{3/2}}\right] \qquad \text{(C-7)}$$

$$= \frac{\mu_0 q_{\mathrm{m}}^2}{2\pi}\left(\frac{1}{2h}\right)^2\left[1 - \frac{1}{\left(1+\left(\frac{\delta}{2h}\right)^2\right)^{3/2}}\right] \qquad \text{(C-8)}$$

$$= \frac{3\mu_0 m^2}{64\pi h^4}.$$

The equilibrium condition $F_z' - M_0 g = 0$ gives the same equilibrium position h_0 as in Eq. (C-6),

$$\Rightarrow \quad \boxed{h_0 = \left[\frac{3\mu_0 m^2}{64\pi M_0 g}\right]^{\frac{1}{4}}.}$$

C.2

The oscillation frequency about the equilibrium is obtained from

$$F_z' \approx M_0 + \frac{dF_z'}{dz}\Delta z, \qquad \text{(C-9)}$$

where $\Delta z = z - h_0$.

And from $\frac{dF_z'}{dz} = -k = -M_0\Omega^2$ (C-10)

we have

$$k = -\frac{d}{dz}\frac{3\mu_0 m^2}{64\pi h^4} = \frac{3\mu_0 m^2}{16\pi h_0^5} = \frac{4}{h_0}\frac{3\mu_0 m^2}{64\pi h_0^4} = \frac{4M_0 g}{h_0} = M_0\Omega^2 \qquad \text{(C-11)}$$

The angular oscillation frequency

$$\boxed{\Omega = \sqrt{\frac{4g}{h_0}}.} \qquad \text{(C-12)}$$

C.3

$$h_0 = \left[\frac{3\mu_0\left(\frac{4}{3}\pi R^3 M\right)^2}{64\pi\left(\frac{4}{3}\pi R^3\rho_0 g\right)}\right]^{1/4} = \left[\frac{R^3 M^2\mu_0}{16\rho_0 g}\right]^{1/4} \qquad \text{(C-13)}$$

$$\boxed{h_0 = \left[\frac{10^{-18}\times 75^2\times 10^{-4}}{16\times 7400\times 9.8\times\mu_0}\right]^{1/4}\text{m} = 25.\,\mu\text{m}.} \qquad \text{(C-14)}$$

C.4 $$\boxed{\Omega = \sqrt{\frac{4g}{h_0}} = \sqrt{\frac{4\times 9.8}{30\times 10^{-6}}}\ \text{s}^{-1} = 1.3\ \text{kHz}.} \qquad \text{(C-15)}$$